The PHYSICAL WORLD
S207

Maths Handbook

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Preface

Physics is one of the natural sciences which describes the Universe around us. Although we may begin to understand the Universe qualitatively, a more complete study requires a quantitative approach. It then becomes necessary to reach into the toolbox of mathematics to find ways of expressing and exploring the relationships between various physical quantities. For that reason, a number of essential mathematical tools are introduced, developed and used in S207 *The Physical World*, and this *Maths Handbook* is a guide to the mathematics utilized in the course. However, it is important to keep things in perspective, and to remember that the focus of S207 is physics, not mathematics.

There are several ways in which you can use the *Maths Handbook*. Firstly, and especially if you are worried about the mathematics in *The Physical World*, you might like to use the early sections of the handbook to revise some of the fundamental mathematical techniques which are used throughout S207, for example rearranging algebraic equations. If you have time, you may find it helpful to work through the first part of the *Maths Handbook* (up to about Section 9) before commencing your study of the rest of the course. Virtually all of the mathematics covered in the *Maths Handbook* is also taught in the course itself; typically a mathematical technique will be introduced at the point at which you need to use it. However, of necessity, the mathematics in the main part of the course is taught quite quickly. Thus the second use you might like to make of the *Maths Handbook* is as a way of learning new mathematical skills at a slower pace, either before you commence your study of the rest of S207, or alongside it. The *Maths Handbook* gives detailed working for many examples, most of which are taken from S207, with a page reference to the main course given as a marginal note. There are also questions for you to try for yourself, with solutions given in Section 20.

The final use which we recommend for the *Maths Handbook* is as a reference tool throughout the course, to check particular mathematical results or techniques. At the end of Section 20 there is a list of commonly used symbols, units and conversions and a tabulation of the Greek alphabet, which is frequently used in physics and mathematics.

This *Maths Handbook* has been written especially to address the needs of Open University students studying S207 *The Physical World*. However, parts are based on earlier guides for S207 and its predecessor course, S271 *Discovering Physics*, the authors of which are gratefully acknowledged.

Assumed knowledge

The *Maths Handbook* assumes that you can:

- add, subtract, multiply and divide simple numbers, including negative numbers;
- calculate fractions and decimals;
- use a calculator for simple arithmetic;
- use degrees to measure angles.

If you feel unsure of your ability to do any of these things you should refer to a suitable mathematics textbook. We recommend one of the following:

The 'Maths Help' Section of *The Sciences Good Study Guide*, by A. Northedge, J. Thomas, A. Lane and A. Peasgood, Open University, 1997. (ISBN 0 7492 3411 3)

Countdown to mathematics, Volume 1, by L. Graham and D. Sargent, Addison Wesley Longman, 1981. (ISBN 0 201 13730 5)

1 Algebra

1.1 The use of symbols

The word 'algebra' is used to describe the process of using symbols, usually letters, to represent quantities and the relationships between them. Algebra is a powerful shorthand that enables us to describe physical quantities precisely without having to know their numerical values.

The symbol chosen to represent something is often the first letter of the quantity in question, e.g. m for mass, t for time and l for length, but it isn't always that simple! Greek

letters are also frequently used as symbols, e.g. λ (lambda) for wavelength. A list of Greek letters and their pronunciation is given at the end of this handbook and you will soon become familiar with those that are commonly used. In a sense it doesn't matter which symbol you use to represent a quantity, since the symbol is only an arbitrarily chosen label. For instance, Einstein's famous equation is usually written as '$E = mc^2$' where E is energy, m is mass and c is the speed of light, but the equation could equally well be written using any symbols you want to use, e.g. $q = \alpha S^2$, provided it was made clear that q is used to represent energy, α is used to represent mass and S is used to represent the speed of light. Whilst it is true that the physics and mathematics of *The Physical World* could be presented using a completely different set of symbols, and that each student studying the course could do so using his or her own set of symbols, this handbook and the rest of the course follow convention as far as possible and use symbols that are more generally used. Sometimes the reason for a choice of symbol will be obvious but unfortunately this is not always the case. You will not be required to remember standard equations and constants for the S207 examination (you will be provided with a list of these), but the symbols used to represent constants and quantities are not defined in the list, so it is important that you become familiar with them.

Sometimes a subscript is used to make the meaning of a symbol more specific, e.g. m_e is used for the mass of an electron, υ_x for velocity along the x-axis and T_i for initial temperature. Note that although υ_x, for example, uses two letters, it represents a *single* physical entity; υ_x is *not* the same as υx. It is particularly important to make this distinction because, when using symbols instead of words or numbers, it is conventional to drop the '×' sign for multiplication, so m (for mass) times a (for magnitude of acceleration) is usually written as ma rather than $m \times a$. Thus υx would mean υ multiplied by x, so when writing υ_x, care needs to be taken to keep the subscript small enough and low enough to be read as a subscript, and in algebraic manipulation the υ_x should always be treated as a single symbol.

There are a number of other occasions when a combination of symbols is used to represent a single entity, in particular:

- The symbol Δ (the Greek upper case delta) is frequently used to represent the change in something, so ΔT represents the change in temperature T, not Δ times T.
- Later in this handbook you will be introduced to notation of the form $x = f(t)$. This means that x is a *function* of t, in other words the value of x depends on the value of t in some particular way. It does not mean that x is f multiplied by t.
- Some letters are frequently used in combination with others in particular abbreviations (e.g. sin for sine, cos for cosine).

A few letters have more than one conventional meaning, for example, L is used to mean length, magnitude of angular momentum, and inductance. Other letters have two meanings but lower case is used for one meaning and upper case for the other, for example; υ for speed and V for volume or t for time and T for temperature. Care needs to be taken, but the intended meaning should be clear from the context.

A final possible source of confusion stems from the fact that the same letter may sometimes be used to represent both a physical quantity and a unit of measurement. For example, an object with a mass of 6 kilograms and a length of 2 metres might be described by the relations $m = 6\,\mathrm{kg}$, $l = 2\,\mathrm{m}$, where the letter m is used to represent both mass and the units of length, metres. In all material for this course, and in most other printed text, letters used to represent physical quantities are printed in *italics*, whereas those used for units are not. Units are discussed further in Section 2 of this handbook.

1.2 Mathematical operations and the use of brackets

Physical quantities can be combined using the standard operations of addition (+), subtraction (−), multiplication (×) or division (÷ or /). The order of addition and the order of multiplication are unimportant, i.e. $a + b = b + a$ and $a \times b = b \times a$ (note that this would more usually be written as $ab = ba$). The correctness of these statements can be verified by checking them with numbers replacing the symbols: $2 + 3 = 3 + 2$ and $2 \times 3 = 3 \times 2$.

When more than one mathematical operation is to be performed on a set of numbers or quantities, the conventions of precedence indicate the order in which the operations should be performed.

> In the absence of indications to the contrary, multiplications and divisions should be completed before additions and subtractions.

For example, in the expression 3 + 6/3, the 'six divided by three' is evaluated first to give

$$3+\frac{6}{3}=3+2=5.$$

If the expression 3 + 6/3 had simply been evaluated from left to right, then the 'three plus six' would have been evaluated first and the answer would have been 9/3, i.e. 3 — this is *wrong*!

Where the normal rules of precedence do not make the order clear enough, brackets can be used to show the order of the steps in a calculation. Brackets are used to indicate a grouping of numbers and/or symbols that can be treated as a whole, thus in (3 + 6)/3, the brackets indicate that all of (3 + 6) is divided by 3, and in $A(x + y)$, the brackets indicate that all of $(x + y)$ is multiplied by A. The following rule can be useful:

Operations inside brackets are evaluated before those outside the brackets.

Thus (3 + 6)/3 = 9/3 = 3.

Sometimes it is desirable to 'multiply out' brackets rather than to evaluate them as written. In multiplying out brackets, *each* quantity inside the bracket must be multiplied. Thus we could work out 5(2 + 3) in two different ways:

either $\quad 5(2 + 3) = 5 \times 5 = 25$ (evaluating the expression in the bracket first)

or $\quad 5(2 + 3) = (5 \times 2) + (5 \times 3) = 10 + 15 = 25$ (multiplying out first).

Likewise, (3+6)/3 could be multiplied out to give 3/3 + 6/3 = 1 + 2 = 3.

Algebraic symbols are manipulated in the same way as pure numbers. Thus

$$3(x - 2y) = 3x - 6y$$

and

$$A(x + y) = Ax + Ay.$$

Remember to keep track of minus signs:

$$-3(x + y) = -3x - 3y$$

because both x and y are multiplied by -3, and

$$-2(a - 4b) = -2a + 8b$$

(remembering that multiplying two negative quantities together gives a positive quantity).

You will not always be trying to get rid of brackets; sometimes an expression is more useful (or just tidier) if the common factor in all the terms is written separately, outside a bracket. So, for example, you may choose to write $Ax + Ay$ as $A(x + y)$.

The usefulness of multiplying out and then reintroducing brackets can be seen in the following worked example, which is taken from *Describing motion* page 18.

DM page 18

Example 1.1 You need to simplify the expression

$$\frac{(At_2+B)-(At_1+B)}{t_2-t_1}.$$

First of all remove the brackets (remembering that the minus before the second bracket means that when the bracket is removed everything from within that bracket will change sign). Then simplify the expression. This gives

$$\frac{(At_2+B)-(At_1+B)}{t_2-t_1}=\frac{At_2+B-At_1-B}{t_2-t_1}$$
$$=\frac{At_2-At_1}{t_2-t_1}.$$

Now note that, in the numerator (the top line of the fraction), both t_2 and t_1 are multiplied by A so we can write

$$\frac{At_2-At_1}{t_2-t_1}=\frac{A(t_2-t_1)}{t_2-t_1}.$$

There is a $(t_2 - t_1)$ in both the numerator and denominator (bottom line) of the fraction, so we can cancel, i.e. divide both numerator and denominator by $(t_2 - t_1)$, to give

$$\frac{A(t_2 - t_1)}{t_2 - t_1} = A. \quad \blacksquare$$

To multiply one bracket by another, multiply each term in the second bracket by each term in the first bracket. So

$$(a + b)(c + d) = \{a(c + d)\} + \{b(c + d)\} = ac + ad + bc + bd$$

Example 1.2

$$\begin{aligned}(a+b)(a-b) &= \{a(a-b)\}+\{b(a-b)\} \\ &= a^2 - ab + ba - b^2 \\ &= a^2 - b^2. \quad \blacksquare\end{aligned}$$

Exactly the same rule is followed to work out the square of an expression in brackets:

Example 1.3

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b) \\ &= \{a(a+b)\}+\{b(a+b)\} \\ &= a^2 + ab + ba + b^2 \\ &= a^2 + 2ab + b^2. \quad \blacksquare\end{aligned}$$

Example 1.4

$$\begin{aligned}(a-b)^2 &= (a-b)(a-b) \\ &= a^2 - ab - ba + b^2 \\ &= a^2 - 2ab + b^2. \quad \blacksquare\end{aligned}$$

The results in Examples 1.2, 1.3 and 1.4 are particularly useful and worth remembering:

$$\boxed{\begin{aligned}&a^2 - b^2 = (a+b)(a-b) \\ &(a+b)^2 = a^2 + 2ab + b^2 \\ &(a-b)^2 = a^2 - 2ab + b^2.\end{aligned}}$$

You should take great care over the positioning of brackets. Note, for example, that the expressions $\left(\frac{1}{2}mv\right)^2$, $\frac{1}{2}(mv)^2$ and $\frac{1}{2}mv^2$ have very different meanings; in the first expression the $\frac{1}{2}$, the m and the v should all be squared to give $\frac{1}{4}m^2v^2$, in the second expression the m and the v should be squared but not the $\frac{1}{2}$, to give $\frac{1}{2}m^2v^2$, and in the third expression just the v should be squared. Furthermore, the square root sign and the horizontal line used to indicate division can both be thought of as containing invisible brackets, i.e. the square root is taken to apply to everything within the sign and the division applies to everything above and below the line. So in

$$T = \frac{u_y + \sqrt{u_y^2 + 2gh}}{g}$$ (from *Describing motion* page 75)

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the square root applies to the whole of $(u_y^2 + 2gh)$ and all of $(u_y + \sqrt{u_y^2 + 2gh})$ should be divided by g.

The following questions are intended to test your understanding of the use of brackets. Answers to these and all similar questions are given at the end of the *Maths Handbook*, but you are strongly encouraged to have a go yourself before looking at the answers.

Question 1 Multiply out the following expressions to eliminate the brackets:

(i) $\frac{1}{2}(v_x + u_x)t$, (ii) $(a - 2b)^2$. $\blacksquare$

Question 2 Simplify the following:

(i) $\dfrac{(a-b)-(a-c)}{2}$

(ii) $\dfrac{a^2-b^2}{a+b}$. (*Hint*: look at Example 1.2.) ■

Question 3 Which of the following expressions are equivalent?

(i) $4a$, (ii) $\sqrt{4a^2}$, (iii) $4\sqrt{a^2}$,

(iv) $2(a+a)$, (v) $\dfrac{2a+6a}{2}$, (vi) $2a+\dfrac{6a}{2}$. ■

1.3 Manipulating fractions

Fractions can be written in several different ways, thus $1 \div 2$, $1/2$ and $\frac{1}{2}$ all mean the same thing (one divided by two, i.e. a half). More generally

$$a \div b = a/b = \frac{a}{b}.$$

It is worth further emphasizing one equality, namely that half of anything is the same as that entity divided by two. Thus

$$\frac{1}{2}mv^2 = \frac{mv^2}{2} = mv^2/2.$$

Algebraic fractions are manipulated in exactly the same way as numerical fractions.

(i) *Multiplication*: To multiply a fraction by a quantity which is not a fraction, multiply the numerator (top line) of the fraction by the quantity, for example:

$$\frac{3}{7}\times 2 = \frac{3\times 2}{7} = \frac{6}{7} \text{ so, } \boxed{\frac{a}{b}\times c = \frac{ac}{b}.}$$

To multiply two or more fractions, multiply the numerators (top lines) together and also multiply the denominators (bottom lines) together.

$$\frac{2}{5}\times\frac{3}{4} = \frac{6}{20} = \frac{3}{10} \text{ so, } \boxed{\frac{a}{b}\times\frac{c}{d} = \frac{ac}{bd}.}$$

(ii) *Division*: To divide a fraction by another quantity, multiply the denominator of the fraction by the quantity you are dividing by.

$$\frac{1}{2}\div 4 = \frac{1}{2\times 4} = \frac{1}{8} \text{ so, } \boxed{\frac{a}{b}\div c = \frac{a}{bc}.}$$

To divide by a fraction, multiply by its reciprocal (i.e. by the fraction turned upside down)

$$\frac{1}{3}\div\frac{1}{6} = \frac{1}{3}\times\frac{6}{1} = 2 \text{ so, } \boxed{\frac{a/b}{c/d} = \frac{a}{b}\div\frac{c}{d} = \frac{a}{b}\times\frac{d}{c} = \frac{ad}{bc}.}$$

(iii) *Addition and subtraction*: In order to add or subtract two fractions, it is necessary for them both to have the same denominator, otherwise you are not adding similar quantities; it would be like saying 3 apples + 2 oranges = 3 bananas, which is clearly nonsense! In numerical work, it is usually convenient to pick the smallest possible number for this denominator (the '*lowest common denominator*'). For example:

$$\frac{1}{3}-\frac{1}{6} = \frac{2}{6}-\frac{1}{6} = \frac{2-1}{6} = \frac{1}{6}.$$

If the lowest common denominator is not easy to spot, it is perfectly acceptable to use *any* common denominator. Perhaps it is easiest to multiply the top and bottom of the first fraction by the denominator of the second fraction, and the top and bottom of the second fraction by the denominator of the first. An example may make this clearer:

$$\frac{1}{3}-\frac{1}{6}=\frac{1\times 6}{3\times 6}-\frac{1\times 3}{6\times 3}$$
$$=\frac{6}{18}-\frac{3}{18}$$
$$=\frac{3}{18}$$
$$=\frac{1}{6}.$$

This is the method to apply to algebraic fractions:

$$\frac{1}{a}+\frac{1}{b}=\frac{b}{ab}+\frac{a}{ab}=\frac{b+a}{ab}$$
$$\frac{1}{a}-\frac{1}{b}=\frac{b}{ab}-\frac{a}{ab}=\frac{b-a}{ab}.$$

Example 1.5 *Describing motion* page 38 gives the equation

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$$s_x=u_x\frac{(\upsilon_x-u_x)}{a_x}+\frac{1}{2}\frac{(\upsilon_x^2-2\upsilon_x u_x+u_x^2)}{a_x}$$

and says that 'after a little algebra', this equation simplifies to

$$s_x=\frac{(\upsilon_x^2-u_x^2)}{2a_x}.$$

Let's work through the 'little algebra'. Note that we need to add the two expressions

$$u_x\frac{(\upsilon_x-u_x)}{a_x}\ \text{and}\ \frac{1}{2}\frac{(\upsilon_x^2-2\upsilon_x u_x+u_x^2)}{a_x},$$

but let's simplify each of the expressions first. Multiplying out the bracket in the first expression gives

$$u_x\frac{(\upsilon_x-u_x)}{a_x}=\frac{u_x(\upsilon_x-u_x)}{a_x}=\frac{u_x\upsilon_x-u_x^2}{a_x}$$

and applying the rules for multiplying fractions to the second expression gives

$$\frac{1}{2}\frac{(\upsilon_x^2-2\upsilon_x u_x+u_x^2)}{a_x}=\frac{1\times(\upsilon_x^2-2\upsilon_x u_x+u_x^2)}{2\times a_x}$$
$$=\frac{\upsilon_x^2-2\upsilon_x u_x+u_x^2}{2a_x}.$$

Now we need to add

$$\frac{u_x\upsilon_x-u_x^2}{a_x}\ \text{and}\ \frac{\upsilon_x^2-2\upsilon_x u_x+u_x^2}{2a_x}.$$

A common denominator of the two expressions is $2a_x$, so in order to add them together we need to multiply the top and bottom of the first expression by 2, i.e.

$$s_x=\frac{u_x\upsilon_x-u_x^2}{a_x}+\frac{\upsilon_x^2-2\upsilon_x u_x+u_x^2}{2a_x}$$
$$=\frac{2(u_x\upsilon_x-u_x^2)}{2a_x}+\frac{\upsilon_x^2-2\upsilon_x u_x+u_x^2}{2a_x}$$
$$=\frac{2u_x\upsilon_x-2u_x^2+\upsilon_x^2-2\upsilon_x u_x+u_x^2}{2a_x}$$
$$=\frac{\upsilon_x^2-u_x^2}{2a_x}$$

as required, since $2u_x\upsilon_x=2\upsilon_x u_x$ and $-2u_x^2+u_x^2=-u_x^2$. ■

A note on cancelling

When we have been using common denominators in the preceding section we have been using an important rule, namely

> If numerator and denominator are both multiplied or divided by the same number, the value of the fraction is unaltered.

This can be expressed algebraically as follows:

$$\boxed{\frac{a\times n}{b\times n}=\frac{a}{b}} \text{ and } \boxed{\frac{a\div m}{b\div m}=\frac{a}{b}.}$$

This rule underpins what you are doing when 'cancelling'. In Example 1.1 we had the expression $\frac{A(t_2-t_1)}{(t_2-t_1)}$. Both the numerator and the denominator contained $(t_2 - t_1)$, so we divided top and bottom ('cancelled') by $(t_2 - t_1)$ to give

$$\frac{A(t_2-t_1)}{(t_2-t_1)}=\frac{A}{1}=A\,.$$

Question 4 Simplify the following expressions:

(a) $h\times\frac{\upsilon}{\lambda}$, (b) $\frac{h\upsilon}{h\lambda}$, (c) $\frac{\mu_0}{2\pi}\times\frac{i_1i_2}{d}$, (d) $\frac{3x}{2t}\Big/2$,

(e) $\frac{2xy}{z}\div\frac{z}{2}$, (f) $\frac{2}{3}+\frac{5}{6}$, (g) $\frac{a}{b}-\frac{c}{d}$. ■

1.4 Rearranging equations

Physical laws are often expressed using algebraic equations, e.g. $E = hf - \phi$ (don't be concerned at this stage with the meaning of the symbols E, h, f and ϕ). This equation would be very useful in the form in which it is given if you had numerical values for h, f and ϕ and were trying to find E. However, if you were trying to find f, say, then it would be more convenient to make f the subject by rearranging the equation. The word *subject* is used here to mean the term written by itself, usually to the left of the equals sign. There are many different ways of teaching people to rearrange equations, and if you are happy with a method which you have learnt previously it is probably best to stick with this method. However, if you have not yet found a way of rearranging equations which suits you, you might like to try the method explained in the box below.

How to rearrange equations

You can think of equations as being like an old-fashioned set of kitchen scales, balanced at the equals sign. You can then apply the following rules

1 Whatever you do mathematically to one side of an equation you must also do to the other.

2 To 'undo' an operation (e.g. +, −, ×, ÷) do the opposite:

So if you want to 'get rid of' an expression which is *added* to the term you want, *subtract* that expression from both sides of the equation.

If you want to 'get rid of' an expression which is *subtracted* from the term you want, *add* that expression to both sides of the equation.

If you want to 'get rid of' an expression which is *multiplied* by the term you want, *divide* both sides of the equation by that expression.

If you want to 'get rid of' an expression by which the term you want has been *divided*, *multiply* both sides of the equation by that expression.

If you are trying to make a term the subject of an equation and you currently have an expression for the *square* of the term, take the *square root* of both sides of the equation.

To get rid of an unwanted *square root*, you should *square* both sides of the equation.

The principles outlined in the box are illustrated in the following examples, all of which are taken from *The Physical World*. Note that the worked examples get progressively more complex, but the underlying principles remain the same throughout. It is very important that you should be able to tackle all of the examples, either by a method you've learnt previously or using the method outlined in the box. You should then be able to rearrange any equation you meet in S207. There are some questions for you to try by yourself at the end the section.

You are not required to *remember* any equations for the S207 examination.

Even if the numerical values of algebraic quantities are known, it is advisable to retain the symbols in any algebraic manipulations until the very last step when all the numerical values can be substituted in. This allows you to see the role of each quantity in the final answer, and it generally minimizes errors too.

Example 1.6 Suppose we want to rearrange the equation

PM page 14

$$F = ma \qquad \text{(from \textit{Predicting motion} page 14)}$$

to obtain an expression for a.

To isolate a we need to get rid of m, and a is currently *multiplied* by m so, according to rule 2 in the box, we need to *divide* by m. Remember (from rule 1) that we must do this to *both sides of the equation*, so we have

$$\frac{F}{m} = \frac{ma}{m}.$$

The m in the numerator (top) of the fraction on the right-hand side cancels with the m in the denominator (bottom) to give

$$\frac{F}{m} = a\text{, i.e. } a = \frac{F}{m}. \quad \blacksquare$$

Example 1.7 Suppose we want to rearrange the equation

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$$\upsilon_x = u_x + a_x t \qquad \text{(from \textit{Describing motion} page 37)}$$

to find an expression for u_x.

To find an expression for u_x we need to get rid of the product $a_x t$, which we can treat as a single expression on this occasion. $a_x t$ is currently added to u_x, so we need to subtract $a_x t$ from both sides of the equation. This gives

$$\upsilon_x - a_x t = u_x + a_x t - a_x t$$

$$\upsilon_x - a_x t = u_x$$

$$u_x = \upsilon_x - a_x t. \quad \blacksquare$$

Example 1.8 Now suppose we want to rearrange the same equation

$$\upsilon_x = u_x + a_x t$$

but this time to find an expression for t.

It is easiest to treat this as a two-step process:

Step 1: Subtract u_x from both sides:

$$\upsilon_x - u_x = u_x + a_x t - u_x$$

$$\upsilon_x - u_x = a_x t$$

Step 2: Divide both sides by a_x:

$$\frac{\upsilon_x - u_x}{a_x} = \frac{a_x t}{a_x}$$

$$t = \frac{\upsilon_x - u_x}{a_x}. \quad \blacksquare$$

RU page 10

Example 1.9 Suppose we want to rearrange the equation

$$h = \tfrac{1}{2} g t^2 \qquad \text{(from } \textit{The restless Universe} \text{ page 10)}$$

to find an expression for t.

First we need to find an expression for t^2, so multiply both sides by 2 to give

$$2h = gt^2$$

and then divide both sides by g to obtain

$$t^2 = \frac{2h}{g}.$$

To obtain an expression for t from this expression for t^2 we need to undo the square by using the inverse operation, i.e. we need to take the square root of both sides, thus

$$t = \sqrt{\frac{2h}{g}}. \quad \blacksquare$$

Strictly this answer should be written as $t = \pm\sqrt{\dfrac{2h}{g}}$, since the square root of a number has two values, one positive and one negative. (Note that $(+2)^2$ and $(-2)^2$ both give 4, so the square root of 4 could be either +2 or −2.) The square root symbol ($\sqrt{\ }$) denotes only the positive square root, hence the need for the '±' sign, which is read as 'plus or minus'. This reflects the mathematics of the problem. Sometimes the physics of the problem allows us to rule out one or other of the two values; for example, we may know that the answer must be positive.

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Example 1.10 Suppose we want to rearrange the equation

$$g = \frac{4R_{\text{Earth}}\pi^2}{T^2} \qquad \text{(from } \textit{Describing motion} \text{ page 104)}$$

to obtain an expression for T.

First we need to find an expression for T^2, so multiply both sides by T^2 to give

$$T^2 g = 4R_{\text{Earth}}\pi^2.$$

Then divide both sides by g to obtain

$$T^2 = \frac{4R_{\text{Earth}}\pi^2}{g}.$$

Now take the square root of both sides to give

$$\begin{aligned} T &= \sqrt{\frac{4R_{\text{Earth}}\pi^2}{g}} \\ &= \sqrt{\frac{4\pi^2 R_{\text{Earth}}}{g}}. \end{aligned}$$

(We are only considering the positive square root since T is a time period which we know must be a positive quantity.)

Since $\sqrt{4\pi^2} = 2\pi$, the answer can also be written as

$$T = 2\pi\sqrt{\frac{R_{\text{Earth}}}{g}}. \quad \blacksquare$$

DM page 114

Example 1.11 Suppose we want to rearrange the equation

$$T = 2\pi\sqrt{\frac{l}{g}} \qquad \text{(from } \textit{Describing motion} \text{ page 114)}$$

to obtain an expression for g.

We are trying to obtain an expression for something which is currently within a square root sign, so the best thing to do first is to get rid of the square root by squaring both sides of the equation:

$$T^2 = \left(2\pi\sqrt{\frac{l}{g}}\right)^2$$

$$= \frac{4\pi^2 l}{g}$$

Now multiply both sides of the equation by g:

$$gT^2 = 4\pi^2 l$$

and divide both sides by T^2 to obtain

$$g = \frac{4\pi^2 l}{T^2}\,.$$

Note that this expression could also be written as

$$g = l\left(\frac{2\pi}{T}\right)^2\,. \quad \blacksquare$$

The following questions are intended to test your skill in rearranging equations.

Question 5 Rearrange each of the following equations to give expressions for the frequency f:

(a) $E = hf$, (b) $E = hf - \phi$. $\blacksquare$

Question 6 Rearrange each of the following equations to give expressions for the mass m:

(a) $E = \dfrac{-GmM}{r}$, (b) $E^2 = p^2c^2 + m^2c^4$, (c) $f = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}}$. $\blacksquare$

1.5 Combining equations

Consider the equation $E = hf$ again; suppose that you know h and are trying to find E. Suppose, however, that you don't know f, but instead know the values of c and λ in a second equation, $c = f\lambda$, which is true for the same f. It would be possible to calculate a value for f from the second equation and then substitute this value in the first equation so as to find E. However, it might be more useful to do the substitution *algebraically*, as in the following example:

Example 1.12 Combine the following two equations to find an expression for E not involving f.

$$E = hf \tag{1.1}$$

$$c = f\lambda. \tag{1.2}$$

First rearrange Equation 1.2 to give

$$f = \frac{c}{\lambda}.$$

Then substitute this expression for f into Equation 1.1 to obtain

$$E = h\times\frac{c}{\lambda} = \frac{hc}{\lambda}\,. \quad \blacksquare$$

This mathematical technique, sometimes referred to as *elimination*, can be used in more complex examples such as Example 1.13, which is taken from pages 38–39 of *Describing motion*.

Example 1.13 Combine the following two equations to find an expression for s_x not involving a_x:

DM page 39

$$\upsilon_x = u_x + a_x t \tag{1.3}$$

$$\upsilon_x^2 = u_x^2 + 2a_x s_x\,. \tag{1.4}$$

First rearrange Equation 1.3 to give an expression for a_x:

$$v_x - u_x = a_x t, \qquad \text{(subtracting } u_x \text{ from both sides)}$$

$$a_x = \frac{v_x - u_x}{t}, \qquad \text{(dividing both sides by } t\text{).}$$

Then substitute this expression into Equation 1.4 to give

$$v_x^2 = u_x^2 + \frac{2(v_x - u_x)s_x}{t}.$$

Then rearrange this to obtain

$$v_x^2 - u_x^2 = \frac{2(v_x - u_x)s_x}{t}$$

$$s_x = \frac{(v_x^2 - u_x^2)t}{2(v_x - u_x)}$$

where we have multiplied both sides by t and divided by $2(v_x - u_x)$.

So $s_x = \dfrac{(v_x + u_x)(v_x - u_x)t}{2(v_x - u_x)}$ (by reference to Example 1.2)

$$s_x = \frac{(v_x + u_x)t}{2} = \frac{1}{2}(v_x + u_x)t.$$

Note that this is Equation 1.28d, given at the top of page 39 of *Describing motion*. ■

In this, and other examples, you may have been able to arrive at the answer in fewer steps. This is acceptable. However, even if you choose to show fewer steps than we have done, we would encourage you to lay out your calculations carefully and to explain your working fully.

Question 7 (a) Combine $E_{\text{trans}} = \frac{1}{2}mv^2$ and $p = mv$ to give an expression for E_{trans} not involving v. (b) Combine $c = f\lambda$ and $n\lambda = d\sin\theta_n$ to give an expression for f not involving λ. ■

1.6 Simultaneous equations

Two different equations containing the same two unknown quantities are called *simultaneous equations* if both equations must be satisfied (hold true) simultaneously. It is possible to solve such equations by using one equation to eliminate one of the unknown quantities from the second equation, in an extension of the method we used in Section 1.5. An example should make the procedure clear.

Example 1.14 Suppose we know that

$$x + y = 7 \qquad (1.5)$$

and

$$2x - y = 2. \qquad (1.6)$$

If we rewrite Equation 1.5 to give an expression for y in terms of x, then we can insert this result into Equation 1.6 to give an expression for x alone.

Rearrangement of Equation 1.5 gives

$$y = 7 - x. \qquad (1.7)$$

Substituting for y in Equation 1.6 then gives

$$2x - (7 - x) = 2$$

$$3x - 7 = 2$$

$$3x = 9$$

$$x = 3.$$

Substitution for x into Equation 1.7 shows that

$$y = 7 - x = 7 - 3 = 4.$$

So the solution, i.e. the values for x and y for which both of the equations hold true, is $x = 3$ and $y = 4$. We could have arrived at the same result by using Equations 1.5 and 1.6 in various different orders, but there is only one correct answer. ■

Note the important fact that in order to find two unknowns, two different equations relating them are required. By extension, *it is always necessary to have as many equations as there are unknowns*. You will find yourself constantly applying this principle as you solve physics problems.

Question 8 Solve the following equations to find the values of a and b:

(i) $a - b = 1$, $a + b = 5$;

(ii) $a^2 - b^2 = 8$, $a + b = 2$. ■

2 Unit (dimensional) analysis

In the preceding section we have encouraged you to use algebra as much as possible, i.e. to use symbols rather than substituting numbers into your working at an early stage. Nevertheless, you will often want to substitute numerical values eventually. S207 is a physics course so most of the numbers will be the values of real physical quantities, and as such they will have *units* attached to them. It would be meaningless to say 'my mass is 60', or 'the table is 1.5 long'; you need to add the units and say 60 *kilograms* or 1.5 *metres*. Physical quantities such as mass and length can be regarded as a *product* of a mass and a unit, as in $m = 60 \times$ kilograms.

Any equations involving physical quantities must have the same units on both sides; it would clearly be nonsensical to write 2 metres = 6 seconds. Just as the numbers on each side of an equation must balance, so must the units. This is the basis of *unit analysis* or *dimensional analysis*. Dimensions in the sense used here are analogous to units: they express the nature of a physical quantity in terms of other quantities that are considered more basic. Thus, we can say that the units of area are metre^2, or equivalently, that area has the dimensions of length^2.

A good habit to cultivate is that of checking, whenever you write an equation, whether the units on either side match — in other words whether the equation is 'dimensionally correct'. For example, suppose you have derived the expression $\sqrt{\dfrac{Ft^2}{a}}$ for a change in energy, where F is the magnitude of a force (measured in newtons, N), t a time (measured in seconds, s) and a the magnitude of an acceleration (measured in metres per second squared, m/s^2). You will learn in *The Physical World* that the units of energy are joules (J) and 1 joule is 1 newton multiplied by 1 metre. Similarly 1 N = 1 kg m/s^2.

A quick check on the units of your expression should be enough to alert you that something is amiss:

The units of energy are J = N m = kg m^2/s^2

The units of $\sqrt{\dfrac{Ft^2}{a}}$ are $\sqrt{\dfrac{\mathrm{N} \times \mathrm{s}^2}{\mathrm{m/s}^2}} = \sqrt{\dfrac{\mathrm{kg\ m/s}^2 \times \mathrm{s}^2}{\mathrm{m/s}^2}} = \sqrt{\mathrm{kg\ s}^2}$.

The units on either side of the proposed 'equation' are not the same, and this is sufficient reason to state unequivocally that the equation is wrong. Note however, that the reverse is not necessarily true. An equation may have the same units on either side (i.e. be dimensionally correct), but still be wrong because of a missing numerical factor or some other error.

Unit analysis can also help if you cannot quite remember an equation. For example, is the equation for the speed of a wave

$\upsilon = f\lambda$ or $\upsilon = f/\lambda$ or $\upsilon = \lambda/f$?

Well, the units of wavelength λ are m, and the units of frequency f are Hz = 1/s. The only way to combine these to get the units of speed, m/s, is to multiply them, so the correct formula is $\upsilon = f\lambda$.

Question 9 Use the fact that the units of mass (m_1, m_2) are kilograms (kg), the units of distance (r) are metres (m), the units of magnitude of force (F) are newtons (N), the units of energy (E) are joules (J), the units of the Universal gravitational constant (G) are N m^2/kg^2, and that

$$1\ \text{N} = 1\ \text{kg m/s}^2 \text{ and } 1\ \text{J} = 1\ \text{N m}$$

to check whether the following equations are dimensionally correct:

(a) $E = m_1 m_2 r$, (b) $F = \dfrac{G m_1 m_2}{r^2}$. ■

3 Powers, roots and reciprocals

A superscript written after a number, such as the 4 in 2^4, indicates the power to which the number (2 in this case) should be raised. In this example, the power (also known as an *index* or *exponent*) indicates repeated multiplication. So $2^4 = 2 \times 2 \times 2 \times 2 = 16$ and can be read as 'two to the power of four', or simply 'two to the four'. Symbols and units of measurements can also be raised to a power. For example, the area of a square of side L is $L \times L = L^2$, and could be measured in square metres, written m^2.

It is very important to understand how to manipulate the indices when quantities are multiplied or divided. As an example, consider multiplying 2^3 by 2^2. This can be written out as

$$2^3 \times 2^2 = (2 \times 2 \times 2) \times (2 \times 2) = 2^5.$$

Generalizing from this example, for any quantity y,

$$y^a \times y^b = y^{a+b}. \tag{3.1}$$

From this rule, we can deduce many other properties of indices.

(i) For example, it shows that $y^0 = 1$. This follows from

$$y^a \times y^0 = y^{(a+0)} = y^a,$$

i.e. multiplying any quantity by y^0 leaves it unchanged. So,

$$y^0 = 1 \text{ for any value of } y. \tag{3.2}$$

(ii) Equation 3.1 can then be used to demonstrate the meaning of a *negative power*.

Since $y^a \times y^{-a} = y^{a-a} = y^0 = 1$, dividing both the left- and right-hand sides of this equation by y^a shows that

$$y^{-a} = \frac{1}{y^a}. \tag{3.3}$$

This is described by saying that y^{-a} is the *reciprocal* of y^a.

For example

$$10^{-2} = \frac{1}{10^2} = \frac{1}{100} = 0.01.$$

Negative powers are frequently used with symbols in the units of physical quantities. For instance, speed is measured in metres per second, written in symbols as m/s or m s^{-1}.

By use of negative indices, Equation 3.1 can easily be applied to situations in which quantities are divided by one another. For example,

$$\frac{10^5}{10^3} = 10^5 \times 10^{-3} = 10^{5-3} = 10^2$$

or more generally,

$$\frac{y^a}{y^b} = y^{a-b}. \tag{3.4}$$

Note that

$$\frac{y^a}{y^{-b}} = y^{a-(-b)} = y^{a+b} = y^a \times y^b.$$

(iii) A *fractional power* denotes the *root* of a number and this too can be deduced from Equation 3.1:

$$y^{1/2} \times y^{1/2} = y^{(1/2+1/2)} = y^1 = y.$$

So $y^{1/2}$ is the quantity that when multiplied by itself gives y in other words, $y^{1/2}$ is a square root of y. By convention we use $y^{1/2}$ to represent the positive square root of y, so

$$y^{1/2} = \sqrt{y}.$$

More generally, the quantity $y^{1/n}$ is the nth root of y,

$$y^{1/n} = \sqrt[n]{y} \tag{3.5}$$

for example $64^{1/3} = \sqrt[3]{64} = 4$.

(iv) Consider raising to some power a quantity that already has an index, such as $(2^2)^3$. Writing this out in full shows that

$$(2^2)^3 = (2^2) \times (2^2) \times (2^2) = 2^{2+2+2} = 2^6 = 2^{2\times 3},$$

or in general

$$(y^a)^b = y^{ab}. \tag{3.6}$$

Like Equation 3.1, this rule applies to any powers, whether positive or negative, integer or fractional.

Thus $x^{3/2} = (x^{1/2})^3 = (\sqrt{x})^3$ and $(27)^{2/3} = (27^{1/3})^2 = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$.

Question 10 Simplify the following to the greatest possible extent. (Try to do so without using a calculator at this stage, but see the note on calculator use at the end of Section 4.)

(a) $10^2 \times 10^3$,

(b) $10^2/10^3$,

(c) t^2/t^{-2},

(d) $\sqrt{10^4}$,

(e) $100^{3/2}$

(f) $(125)^{-1/3}$

(g) $\left(\dfrac{x^4}{4}\right)^{1/2}$

(h) $(2\,\text{kg})^2/(2\,\text{kg})^{-2}$. ■

4 Powers of ten and scientific notation

Powers of ten, such as

$$10^6 = 1000\,000 \text{ (a million) or } 10^{-3} = \frac{1}{1000} = 0.001 \text{ (a thousandth)}$$

very often appear in physics because they provide a shorthand way of writing down very large or very small quantities.

A quantity is said to be in *scientific notation* if its value is written as a decimal number between 1 and 10 multiplied by 10 raised to some power.

For example, the radius of the Earth is 6378 km. In scientific notation this would be written as 6.378×10^3 km or 6.378×10^6 m. Scientific notation is equally useful for very small quantities: for instance, the mass of an electron is conveniently written as 9.109×10^{-31} kg.

Scientific notation is a great aid to calculation, since the decimal numbers and the powers of ten can be dealt with separately. Suppose for example, that we wanted to calculate the value of a light-year (the distance light travels in one year), knowing that the speed of light is $3.00 \times 10^8\,\text{m s}^{-1}$.

$$1 \text{ year} = 365 \text{ days} \times 24 \frac{\text{hours}}{\text{day}} \times 60 \frac{\text{minutes}}{\text{hour}} \times 60 \frac{\text{seconds}}{\text{minute}} \approx 3.154 \times 10^7 \text{ s}.$$

(The ≈ symbol, a wavy '=' sign, is read as 'approximately equal to'. This symbol and others with which you may not be familiar are listed towards the end of this handbook.)

In that time, light travels a distance given by the product of the elapsed time and the speed:

$$\begin{aligned}\text{distance} &= \text{elapsed time} \times \text{speed}\\ &= 3.154\times 10^{7}\ \text{s} \times 3.00\times 10^{8}\ \text{m s}^{-1}\\ &= (3.154\times 3.00)\times(10^{7}\times 10^{8})(\text{s m s}^{-1})\\ &\approx 9.46\times 10^{15}\ \text{m}.\end{aligned}$$

Note that in calculating the answer we have dealt with the decimal numbers and the powers of ten separately. We have also checked that the units make sense in the way described in Section 2. Note too that the final answer is quoted to just 3 significant figures (significant figures are discussed further in Section 6).

Question 11 Express the following numbers in scientific notation

(a) 1467 851, (b) 0.0046, (c) 11×10^{6}, (d) 0.0031×10^{-2}. ■

Question 12 Evaluate the following in scientific notation (you should be able to do this without using a calculator)

(a) $(3 \times 10^{6}) \times (7 \times 10^{-2})$, (b) $\dfrac{8\times 10^{4}}{4\times 10^{-1}}$, (c) $\dfrac{10^{4}\times(4\times 10^{4})}{0.001\times 10^{-2}}$. ■

A note on the use of calculators for powers and scientific notation

In the preceding two sections we have encouraged you to do calculations *without* using a calculator. However, in S207 you will also need to be able to use your calculator to handle powers and scientific notation, and now would be a good time to check that you know how to do this. Scientific calculators vary a little in the way in which powers and scientific notation are input, so you may need to check in your calculator's instruction booklet, but you will probably be using a button labelled 'x^{y}' or '^' for powers and a button labelled 'EE' or 'EXP' for scientific notation. Take particular care when entering numbers such as 10^{4} into your calculator — remember that this is 1×10^{4} *not* 10×10^{4}.

You should also check that you know how to read the display of your calculator when the numbers are shown in scientific notation. For example, try inputting '$(3 \times 10^{9}) \times (2 \times 10^{6})$' into your calculator: the correct answer is 6×10^{15}, but some calculators display the answer in a way which could easily be read as 6^{15}, which is clearly wrong. Take care!

To familiarize yourself with the power and scientific notation functions on your calculator, we suggest that you repeat the numerical parts of Question 10 and Question 12, this time using your calculator, and check that you get the correct answers by this method too.

5 Logarithms

Logarithms are not used very much in S207, and they are not used all until nearly half-way through the course. However, you will need to have an understanding of the important points.

As we discussed in the previous section, numbers such as 100 and 0.01 can be expressed in powers of 10, respectively as 10^{2} and 10^{-2}. In fact, by the use of decimal powers, *any* number can be expressed as a power of ten. For example, you can easily verify the following on your calculator:

$0.1 = 10^{-1}$ $\quad$ $1 = 10^{0}$

$2 \approx 10^{0.301}$ $\quad$ $3 \approx 10^{0.477}$

$3.16 \approx 10^{0.5} = 10^{1/2} = \sqrt{10}$ $\quad$ $10 = 10^{1}$

$251 = 10^{2.4}$.

In each case, the *power* to which 10 is raised is called the *logarithm to base ten* or *common logarithm* (abbreviated $\log_{10}$, log, or sometimes lg) of the resulting number. For example:

$100 = 10^2$ so $\log_{10}100 = 2$

$0.1 = 10^{-1}$ so $\log_{10}0.1 = -1$

$2 \approx 10^{0.301}$ so $\log_{10}2 \approx 0.301$

$251 \approx 10^{2.4}$ so $\log_{10}251 \approx 2.4$.

Taking a logarithm to base 10 is the *inverse* of raising 10 to a power, i.e. the '$\log_{10}$' button on a calculator reverses the operation of the '10^x' button. You can use your calculator to check this for an arbitrarily chosen number, e.g. 5.6; pressing the '10^x' button gives 398 107.1706 and finding the logarithm to base ten of the latter number returns the display to 5.6. More generally,

$$\text{if } x = 10^a \text{ then } \log_{10}x = a \tag{5.1}$$

Question 13 Without using a calculator, write down the value of

(a) $\log_{10}1000$, (b) $\log_{10}0.001$, (c) $\log_{10}\sqrt{10}$. ■

Logarithms are useful for dealing with numbers that range from very large to very small. Figure 1 (overleaf) illustrates a range of lengths in the natural world, most conveniently written in scientific notation and plotted using a *logarithmic scale*. In the figure, each step represents a *multiplication* by a factor of 10^2. Contrast this with the more usual linear scale in which each step represents the *addition* of some constant amount.

Base ten is commonly used for logarithms, since ten is the base of our counting system, but of course any number can be raised to a power and hence used as a base for logarithms. Logarithms to base e (where e is a special number having a value ≈ 2.718), are called *natural logarithms* (abbreviated $\log_e$ or ln), and are used extensively in physics. (The significance of e will become clearer in Section 14 on the *exponential function*.)

$$\text{If } y = \mathrm{e}^b\text{, then } \log_e y = b. \tag{5.2}$$

Note: *Some textbooks use 'log' (without a subscript) to refer to $\log_e$ rather than $\log_{10}$. We recommend writing the subscript, or using 'ln', to avoid ambiguity.*

There is very little manipulation of logarithms in this course, but three important rules (quoted here without proof) will be useful:

$$\log(a \times b) = \log a + \log b \tag{5.3}$$

$$\log\left(\frac{a}{b}\right) = \log a - \log b \tag{5.4}$$

$$\log a^b = b \log a. \tag{5.5}$$

These rules apply irrespective of which base (10, e, or some other) is adopted, and we can demonstrate their use by looking at an example from page 136 of *Classical physics of matter*. This example includes rather complicated symbols, so try to concentrate on the way in which the logarithms are handled.

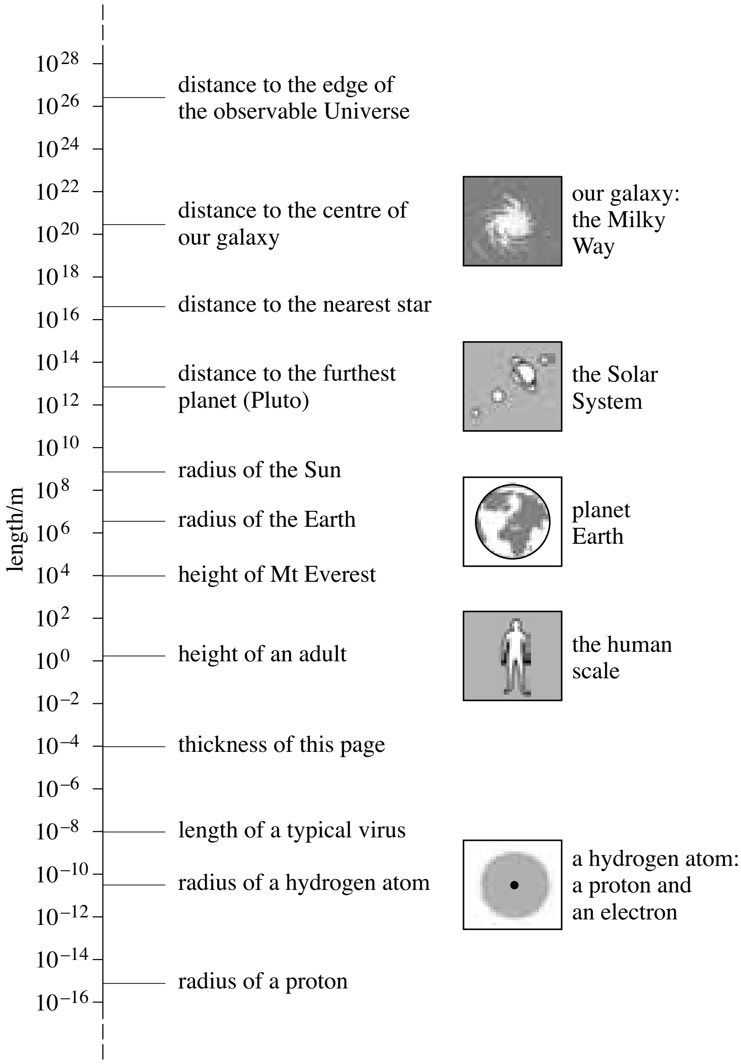

Figure 1 Some examples of lengths. This diagram is plotted using a logarithmic scale, in which each step represents *multiplication* by a factor of 10^2.

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Example 5.1 Show that the expression $C_V \log_e\left(\dfrac{P_2V_2^\gamma}{P_1V_1^\gamma}\right)$ is equal to

$$C_V\left(\log_e\left(\frac{P_2}{P_1}\right) + \gamma\log_e\left(\frac{V_2}{V_1}\right)\right).$$

Starting with the expression in the brackets

$$\frac{P_2V_2^\gamma}{P_1V_1^\gamma} = \frac{P_2}{P_1} \times \frac{V_2^\gamma}{V_1^\gamma}$$

so, from Equation 5.3,

$$\log_e\left(\frac{P_2V_2^\gamma}{P_1V_1^\gamma}\right) = \log_e\left(\frac{P_2}{P_1}\right) + \log_e\left(\frac{V_2^\gamma}{V_1^\gamma}\right).$$

Now $\dfrac{V_2^\gamma}{V_1^\gamma} = \left(\dfrac{V_2}{V_1}\right)^\gamma$ so, from Equation 5.5,

$$\log_e\left(\frac{V_2^\gamma}{V_1^\gamma}\right) = \log_e\left(\frac{V_2}{V_1}\right)^\gamma = \gamma\log_e\left(\frac{V_2}{V_1}\right)$$

i.e. $C_V \log_e\left(\frac{P_2 V_2^\gamma}{P_1 V_1^\gamma}\right) = C_V\left(\log_e\left(\frac{P_2}{P_1}\right) + \gamma \log_e\left(\frac{V_2}{V_1}\right)\right)$. ■

Question 14 Use the general rules of logarithms to show that

$\log_{10}(3x^4) = \log_{10}3 + 4\log_{10}x$. ■

6 Significant figures

An athlete runs a 400 m race and the time taken is recorded as 45.23 seconds. What is the average speed $\langle v \rangle$? (Note that angle brackets are used to denote average values.)

Clearly $\langle v \rangle = (400/45.23)\,\mathrm{m\,s^{-1}}$ If you input this division into your calculator, you will get $8.843\,687\,818\,\mathrm{m\,s^{-1}}$, but given the accuracy of the track-side measurements, there is no justification for retaining more than three or four digits in the answer. The number of digits that you quote when you write down the value of a quantity is known as the number of *significant figures*.

In the case above, the calculator display could be rounded to give

9 (to 1 significant figure)

8.8 (to 2 significant figures)

8.84 (to 3 significant figures)

8.844 (to 4 significant figures)

8.8437 (to 5 significant figures).

If the last significant figure is followed by a digit from 0 to 4 then it is unchanged in rounding; if it is followed by a digit from 5 to 9 then it is increased by one.

Experimental results should always be quoted to a number of significant figures consistent with the precision of the measurement. In general, the value of the last significant figure may be somewhat uncertain, but you should be confident about the values of the other figures.

If two or more quantities are combined, for instance by dividing one by the other, then the result is known only to the same number of significant figures as the *least* precisely known quantity. So if a distance of 399 metres (known to three significant figures) is run in a time of 45.23 seconds (four significant figures) then the value of the speed should be quoted to *three* significant figures, i.e. speed = $\frac{399\,\mathrm{m}}{45.23\,\mathrm{s}} \approx 8.82\,\mathrm{m\,s^{-1}}$.

If a calculation involves multiple steps, it is best to retain more digits through the calculation, and to round to the correct number of significant figures only at the final stage. In this way you will avoid the introduction of rounding errors.

So, how many significant figures should we quote in giving the athlete's average speed? Clearly, the run has been timed to 4 significant figures, but the distance itself is more ambiguous. Does 400 m correspond to 4×10^2 m, i.e. one significant figure, 4.00×10^2 m, i.e. 3 significant figures, or 4.0000×10^2 m, i.e. 5 significant figures? In this context, giving three significant figures seems the most sensible, implying as it does that the distance actually covered is 400 m to within ±0.5 m (which is reasonable given that the athlete will not run straight down the middle of his lane), so the final result for $\langle v \rangle$ is probably best quoted to three significant figures too, i.e. as $8.84\,\mathrm{m\,s^{-1}}$. You should notice that a certain amount of interpretation was necessary in order to arrive at this conclusion, and the appropriate precision to give the result could still be open to further discussion. The physics of the problem, i.e. the shape and length of the track over which the athlete ran, rather than a simple mathematical calculation, determines the correct answer.

Some physics texts adopt the convention that in contrived examples, data can be assumed to have arbitrarily high precision, and that all zeroes in provided numbers such as 6000 are significant. However, you should always think about the context and the physical realities of the situation before making assumptions about the precision to which data are quoted.

Using scientific notation eliminates any doubt over the number of significant figures intended. If the speed of light is quoted as $3 \times 10^8\,\mathrm{m\,s^{-1}}$ or $3 \times 10^5\,\mathrm{km\,s^{-1}}$ then it is clear

that it is being given to just one significant figure. Similarly, if it is quoted as $3.00 \times 10^8\,\mathrm{m\,s^{-1}}$ or $3.00 \times 10^5\,\mathrm{km\,s^{-1}}$ then we know that it is being given to 3 significant figures. In contrast, writing that the speed of light is $300\,000\,\mathrm{km\,s^{-1}}$ could be taken as implying that all six digits are significant. In fact, to such precision the speed of light is $299\,792\,\mathrm{km\,s^{-1}}$.

Now consider significance in small numbers. Suppose you measure the thickness of the spine of a book as 1.6 cm, and you need to convert this to metres. Clearly, you want your result in metres to have the same precision, i.e. the same number of significant figures (two in this case), as your initial measurement. In decimal notation, you would express the result as 0.016 m: *leading zeroes do not count as significant figures*. However, using scientific notation again removes any ambiguity — the thickness of the book is $1.6 \times 10^{-2}\,\mathrm{m}$, again to two significant figures.

Question 15 What, in $\mathrm{km\,s^{-1}}$ and to an appropriate number of significant figures, are the speeds involved in the following situations:

(a) A lorry travels 1.2 km in 65.6 s.

(b) A car travels 0.09 km in 5.1 s.

(c) Light is measured as travelling 3000 km in 0.01 s. ■

When adding or subtracting numbers however, it is often not reasonable to quote the answer to as many significant figures as quoted in the question. For example, I have a strip of paper which is 1.859 m long (that is, I know its length to the nearest millimetre). I cut off a strip of length 1.853 m. What is the length of the piece I have left? The answer must be $(1.859 - 1.853)\,\mathrm{m} = 0.006\,\mathrm{m}$ or $6 \times 10^{-3}\,\mathrm{m}$ which is known to only one significant figure, that is, as before, to the nearest millimetre.

7 Angular measure

Angles are often measured in *degrees*, where 360° equals one complete turn. However, in physics they are frequently and more usefully measured in *radians*. If an arc (i.e. part of the circumference of a circle) of length *s subtends* an angle θ at the centre of a circle of radius r as shown in Figure 2, then

$$\theta\,(\text{in radians}) = \frac{s}{r}. \tag{7.1}$$

An arc of length $2\pi r$ (i.e. the whole circumference) subtends an angle (in radians) of $\dfrac{2\pi r}{r} = 2\pi$, that is,

$$2\pi\ \text{radians} = 360°. \tag{7.2}$$

So 1 radian $= \dfrac{360°}{2\pi} \approx 57.3°$.

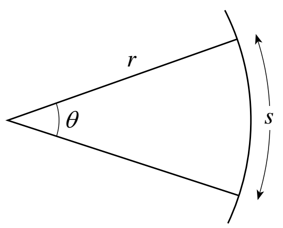

Figure 2 An arc of length s subtending an angle θ at the centre of a circle of radius r.

Question 16 (a) Convert the following to radians, expressing your answer as a fraction/multiple of π: (i) 90°, (ii) 30°, (iii) 180°.

(b) Convert the following to degrees: (i) $\dfrac{\pi}{8}$ radians, (ii) $\dfrac{3\pi}{2}$ radians. ■

8 Triangles and trigonometric ratios

You may have met *Pythagoras*' *theorem* before. This is commonly stated as 'the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides'. In terms of the symbols given in Figure 3, it then follows that

$$h^2 = o^2 + a^2 \quad \text{or} \quad h = \sqrt{o^2 + a^2}.$$

Pythagoras' theorem gives us a way of calculating the length of a third side of a right-angled triangle from a knowledge of the lengths of the other two sides.

The trigonometric ratios, sine, cosine and tangent, relate the ratios of the sides of a right-angled triangle to its acute (<90°) angles, and thus give us a way of calculating the length of a side from knowledge of the length of another side and one of the acute angles, or a way of calculating the acute angles from knowledge of the lengths of two of the sides. Knowledge of one of the acute angles of a right-angled triangle automatically gives the other angle too, since the angles of a triangle always add up to exactly π radians or 180°.

The underlying principle is that any triangle with given angles will have a fixed shape (although it might be of any size). For a right-angled triangle, the ratios of the lengths of any two sides of the right-angled triangle are defined and given the following names:

$$\frac{\text{side opposite an angle}}{\text{hypotenuse}} = \text{sine of the angle} \qquad \sin\theta = \frac{o}{h} \tag{8.1}$$

$$\frac{\text{side adjacent to an angle}}{\text{hypotenuse}} = \text{cosine of the angle} \qquad \cos\theta = \frac{a}{h} \tag{8.2}$$

$$\frac{\text{side opposite an angle}}{\text{side adjacent to an angle}} = \text{tangent of the angle} \qquad \tan\theta = \frac{o}{a} \tag{8.3}$$

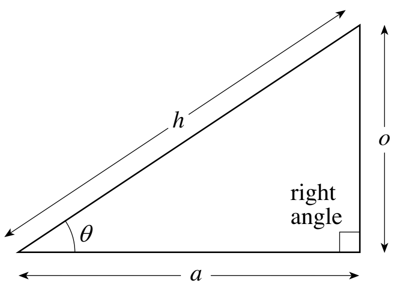

Figure 3 A right-angled triangle. The side opposite the right angle, called the *hypotenuse*, is of length *h*. The side opposite to angle *θ* has length *o*, the side adjacent to it has length *a*.

An example, modified from page 51 of *Describing motion* may help to show the usefulness of the trigonometric ratios:

Example 8.1 Write down expressions for x and y (as shown in Figure 4) in terms of r and θ. Calculate x and y for $r = 6.5$ m and $\theta = 35°$. *DM* page 51

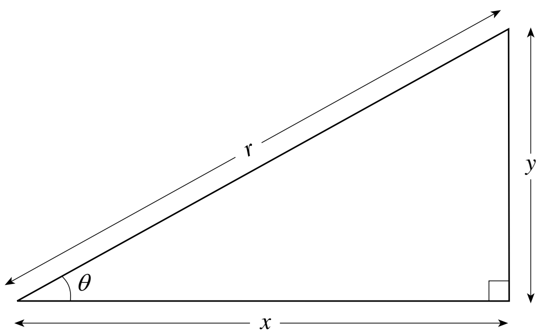

Figure 4 A right-angled triangle. For use in Example 8.1.

The cosine ratio tells us that $\cos\theta = \dfrac{\text{side adjacent to angle}}{\text{hypotenuse}} = \dfrac{x}{r}$, so (multiplying both sides by r) $x = r\cos\theta$.

Similarly, the sine ratio tells us that $\sin\theta = \dfrac{\text{side opposite to angle}}{\text{hypotenuse}} = \dfrac{y}{r}$, so $y = r\sin\theta$.

Using a scientific calculator, with the mode set to 'degrees' gives the following:

$x = r\cos\theta = 6.5\text{ m} \times \cos 35° = 5.3\text{ m}$, to 2 significant figures,

$y = r\sin\theta = 6.5\text{ m} \times \sin 35° = 3.7\text{ m}$, to 2 significant figures. ■

A note on the use of calculators for trigonometric ratios

You should be able to use your calculator to find the sines, cosines and tangents of given angles, but you need to take care that you have set your calculator up properly so that it is responding to an angle given in degrees or radians as appropriate. Usually this involves selecting 'radian mode' or 'degree mode'; see your calculator handbook for details. You can check quickly which is set by calculating cos(3) or cos(π). If the calculator gives a number close to +1, then it is working in degrees, whereas if it gives a number close to −1, then it is working in radians.

You will also need to be able to use your calculator to find the angle which has a particular sine, cosine or tangent. For example, if you know that $\tan\theta = 3/4$. then what is θ? What you are looking for is known as the *arctangent* or *inverse tangent*, and you need to use a button on your calculator labelled 'arctan' or '$\tan^{-1}$'. Check that you can use your calculator to give the correct answer, which is that arctan(0.75) = 37° = 0.64 radians. Your calculator should also be able to calculate 'arcsin' (also known as '$\sin^{-1}$') and 'arccos' (also known as '$\cos^{-1}$'). In S207 we use both notations (arctan and $\tan^{-1}$, arcsin and $\sin^{-1}$, arccos and $\cos^{-1}$). However it is important to avoid confusion with reciprocals in using the notation $\tan^{-1}$, $\sin^{-1}$ and $\cos^{-1}$. Note that

$$\tan^{-1}x \neq \frac{1}{\tan x},\quad \sin^{-1}x \neq \frac{1}{\sin x}\quad\text{and}\quad \cos^{-1}x \neq \frac{1}{\cos x}.$$

For speed in calculations, you may find it worth remembering the values in Table 1 below.

Table 1 Some important values of sin, cos and tan.

	sin	cos	tan
0°	0	1	0
$\frac{\pi}{6} = 30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4} = 45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3} = 60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2} = 90°$	1	0	∞

Question 17 Find the following

(a) (i) $\sin 49°$, (ii) $\cos\dfrac{\pi}{8}$, (iii) $\tan\dfrac{\pi}{4}$; (b) the angle whose sine is 0.1. ■

Question 18 Find the angle α in the right-angled triangle shown in Figure 5. ■

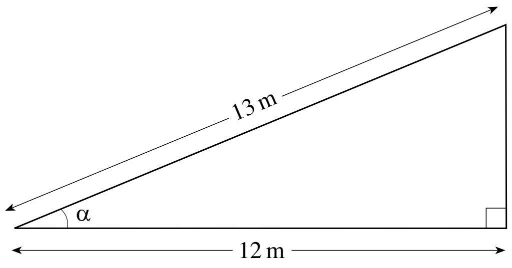

Figure 5 A right-angled triangle. For use in Question 18.

Referring to Figure 3, we see that as the angle θ becomes smaller and smaller, o decreases, and h becomes more and more nearly equal to a as shown in Figure 6. So,

$$\cos\theta = \frac{a}{h} \approx 1 \text{ for small } \theta. \tag{8.4}$$

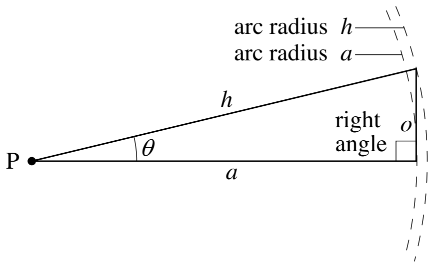

Figure 6 A right-angled triangle with a small angle θ.

Figure 6 also shows that for small θ, the length o approximates the length of an arc of a circle with centre P, and radius a (or h).

The general formula for the length of an arc gives:

For arc at radius a:	For arc at radius h:
arc length = $a\theta$	arc length = $h\theta$
i.e. $o \approx a\theta$	i.e. $o \approx h\theta$
But $\tan\theta = o/a$	and $\sin\theta = o/h$

Therefore,

$$\tan\theta \approx \theta \tag{8.5}$$

and

$$\sin\theta \approx \theta \tag{8.6}$$

where θ is small and in radians.

These so-called *small-angle approximations* hold within 1% accuracy for angles less than about 0.2 radians (11°). Your answer to Question 17b should confirm that $\sin\theta \approx \theta$ when θ is small and in radians.

Additional trigonometric relationships can be derived based on the properties introduced above. By Pythagoras' theorem, and with reference to Figure 3, $o^2 + a^2 = h^2$. Dividing both sides by h^2, we see that

$$\frac{o^2}{h^2} + \frac{a^2}{h^2} = \frac{h^2}{h^2}$$

so $(\sin\theta)^2 + (\cos\theta)^2 = 1$.

$(\sin\theta)^2$ is usually written as $\sin^2\theta$ and $(\cos\theta)^2$ is usually written as $\cos^2\theta$, so

$$\sin^2\theta + \cos^2\theta = 1. \tag{8.7}$$

Also, $\tan\theta = \dfrac{o}{a} = \left(\dfrac{o}{h}\right) \bigg/ \left(\dfrac{a}{h}\right)$, so

$$\tan\theta = \frac{\sin\theta}{\cos\theta}. \tag{8.8}$$

9 Functions and graphs

9.1 Function notation

If the value of one variable, x say, is wholly or partly determined by the value of another, t say, then x is said to be a *function* of t. If we know the precise relationship between x and t then we can represent it by an equation. For instance, the position x of a body moving along a straight line at constant speed υ, starting at x_0, is given by the equation $x = x_0 + \upsilon t$. However, it is also possible to indicate the existence of a relationship between x and t in a very *general* way, writing $x = f(t)$ to indicate that 'x is a function of t'. We then say that t is the *argument* of the function. In the example, we already know the form of f, it is simply $f(t) = x_0 + \upsilon t$, but it is possible to imagine cases where the functional form is different, or even unknown. In these cases, the more general expression $x = f(t)$ can be very useful. Note that there is nothing special about using the letter f to represent the function; we could equally have written $x = g(t)$ where $g(t) = x_0 + \upsilon t$. Also, functions are not limited to situations where x depends on t; we could also, for example, have z as a function of y (e.g. $z = f(y) = Ay^2 + By + C$), V as a function of r (e.g. $V = f(r) = \frac{4}{3}\pi r^3$) or y as a function of θ (e.g. $y = f(\theta) = A\sin\theta$).

The function notation has two great merits

1 Writing $f(t)$ provides a clear visual reminder that f depends on t in a well-defined way.

2 If we want to indicate the value of f that corresponds to a particular value of t, it is easy to do so. For example, the value of $f(t)$ at $t = 2$ can be written $f(2)$.

The only serious disadvantage of the function notation is that you may confuse $f(t)$ with $f \times t$. Be careful!

We will be discussing linear functions of the form $f(t) = At + B$ in more detail in this section and we will introduce others, in particular polynomial, trigonometric and exponential functions, later in the Maths Handbook. Each of these functions can be considered using either its *equation* or its *graph* and we will develop these two concepts in parallel in the following sections.

9.2 A note on graph plotting

There will be many occasions throughout your study of physics when you will need to draw graphs. This subsection gives some guidance for this important activity.

1 Decide which is the *independent variable* and which is the *dependent variable*, and plot the independent variable along the horizontal axis and the dependent variable along the vertical axis. The independent variable is the quantity whose value is chosen by the experimenter or otherwise fixed by the nature of the process, whereas the dependent variable changes as a result of changes made to the independent variable. Thus, if you were asked to plot a graph to show how the position of a car varies with time, you would usually plot time on the horizontal axis and position on the vertical axis because it is the position that varies with time rather than the time that varies with position. This is purely a convention, so if you can't decide which variable is which don't worry too much! The shorthand way of describing a graph in which x is plotted vertically and t horizontally is a graph of '*x against t*' or '*x versus t*'.

2 Fill as much of the graph paper as reasonably possible. You will obtain greater accuracy if the graph is as big as possible. However, take care to use the graph paper sensibly. Graph paper usually has centimetre and millimetre squares, so it is straightforward to use 2 or 5 or 10 divisions on the paper to one physical unit. You should avoid multiples such as 3, 6 and 7.

3 Give the graph a title, e.g. position versus time.

4 Label both axes to show which quantities are being plotted and include the units. Each axis should be plotted as ‘quantity/unit’, e.g. time/s, position/m. Since the quantity includes the unit, the division quantity/unit results in a pure number, and that is what is plotted on the axis.

5 Scale the axes appropriately, especially if the numbers involved are either very large or very small. For example, if the values of time t range from 0 s to 1.0×10^{-5} s, then, rather than plotting t/s and inserting values such as 1.0×10^{-6} s, 2.0×10^{-6} s, etc. along the axis, it is usually more convenient to change the units to microseconds and plot t/μs; the values along the axis will then simply be 1, 2, 3, etc. It is also acceptable to label the axis $t/10^{-6}$ s rather than t/μs if you prefer.

6 Plot the points carefully using crosses or dots within circles.

7 Draw a straight line or curve that best represents the points plotted, i.e. a ‘best fit’ curve. Don’t worry if this doesn’t go through all (or even any!) of the data points, however in drawing a best fit line or curve by eye it should go as close as possible to as many as possible of the points (after excluding or re-plotting any ‘rogue’ points).

Table 2 gives some data showing how the depth of water in each of two paddling pools varies as an increasing volume of water is added. We have plotted the data for Pool 1 for you in Figure 7, and Question 19 asks you to plot the corresponding data for Pool 2.

Table 2 The depth of water, d_1 in Pool 1 and d_2 in Pool 2, corresponding to a volume of water, V, pumped into each pool.

Volume of water, V/m^3	Depth of water in Pool 1, d_1/m	Depth of water in Pool 2, d_2/m
0	0	0.20
100	0.21	0.30
200	0.39	0.40
300	0.59	0.50
400	0.80	0.61
500	1.01	0.69

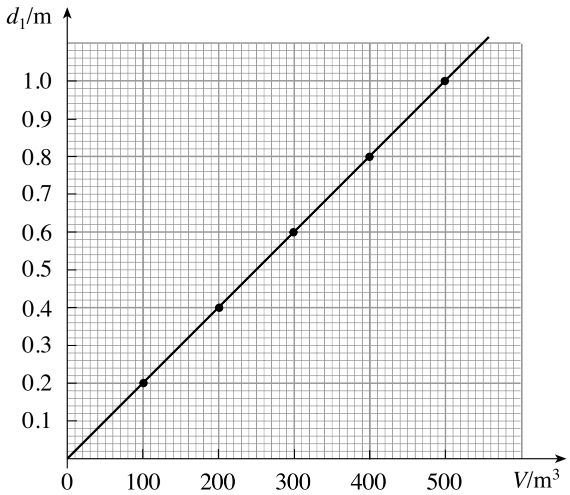

Figure 7 A graph showing how the depth of water, d_1, in Pool 1 varies with the volume of water, V, that has been pumped into the pool.

Question 19 Plot (using the graph paper provided in Figure 8) a graph to show how the depth of water in Pool 2 varies with the volume of water that has been pumped into the pool. Make sure you include all the correct labelling. ■

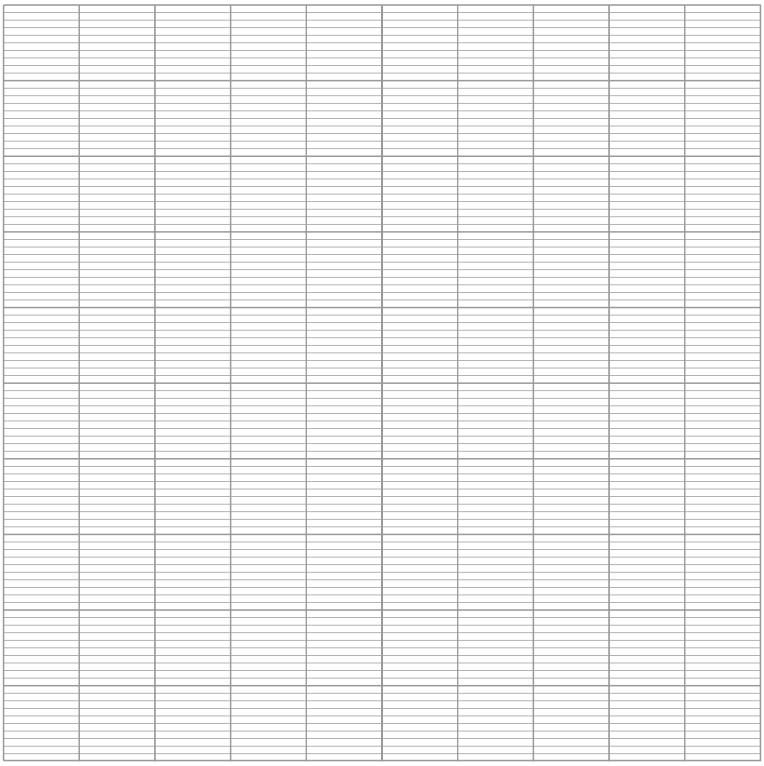

Figure 8 Graph paper for use in Question 19.

9.3 The gradient of a straight-line graph

Any linear (straight-line) graph has a constant slope or *gradient*. Figure 9 shows how to calculate the gradient of the graph we plotted for Paddling Pool 1.

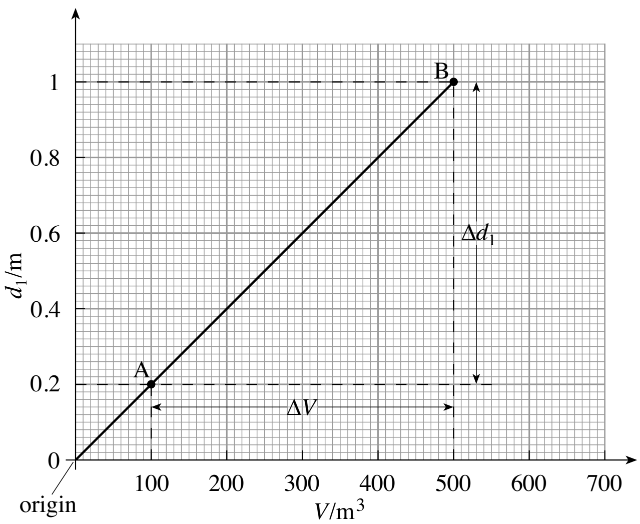

Figure 9 A graph showing how the depth of water, d_1, in Pool 1 varies with the volume of water, V, that has been pumped into the pool. For use in calculating gradient.

First, choose two convenient but well separated points ‘A’ and ‘B’ on the graph, and read off the corresponding values of V, to be called V_A and V_B. The difference between them is

the change in V. Changes are usually denoted by the Greek symbol Δ, thus ΔV (read as 'delta vee') means the change in V

$$\Delta V = V_B - V_A.$$

The corresponding change in d_1, written Δd_1 can also be read off the graph

$$\Delta d_1 = d_{1B} - d_{1A}.$$

The gradient of the line is then defined as

$$\text{gradient} = \frac{\Delta d_1}{\Delta V} = \frac{d_{1B} - d_{1A}}{V_B - V_A}.$$

More generally, for a graph of y versus x,

$$\text{gradient} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}. \qquad (9.1)$$

A convenient way of remembering this is to think of the change in the vertical quantity as the 'rise' and the corresponding change in the horizontal quantity as the 'run', so

$$\text{gradient} = \frac{\text{rise}}{\text{run}}.$$

The gradient of a graph represents the rate of change of one quantity as another quantity changes.

Example 9.1 Find the gradient of the graph shown in Figure 9.

In this case

$$\Delta d_1 = 1.0\,\text{m} - 0.2\,\text{m} = 0.8\,\text{m}$$

$$\Delta V = 500\,\text{m}^3 - 100\,\text{m}^3 = 400\,\text{m}^3$$

$$\text{so gradient} = \frac{\Delta d_1}{\Delta V} = \frac{0.8\,\text{m}}{400\,\text{m}^3} = 0.002\,\text{m}^{-2}.$$

The gradient $\Delta d_1/\Delta V$ represents the rate of increase of depth with volume, which is 0.002 metres (i.e. 2 millimetres) of depth per cubic metre of volume. ■

In this example, the graph slopes upwards from left to right, because *d increases* with V, and so the gradient $\Delta d/\Delta V$ *is positive*. A graph sloping downwards from left to right tells us that one quantity *decreases* as the other increases, and such a graph will have *negative* gradient.

Question 20 Find the gradient of the graph you drew in Question 19. ■

9.4 Straight-line graphs passing through the origin: proportionality

Two quantities related in such a way that if one is doubled the other also doubles are said to be *directly proportional*.

Let us return to the example of Paddling Pool 1. The pool was empty to start with, and we're assuming that the pool has straight sides, so common sense tells us that the depth d of the water in the pool will be directly proportional to the volume V of water that has been added. Looking at Table 2 confirms that, to within measurement uncertainties, if we double V, then d doubles, if we treble V then d trebles, and when $V = 0$, $d = 0$. Thus d is directly proportional to V. This is written as:

$$d \propto V$$

which implies that

$$d = kV$$

where k is a constant called the *constant of proportionality*.

Note from Figure 7 that the graph of d against V is a straight line that passes through the origin (the point where $d = 0$ and $V = 0$). This is a general and very important result. In fact, for any two variables x and y which are directly proportional

- we can write $y = kx$ where k is a constant,
- a graph of y against x will be a straight line passing through the origin,
- the constant of proportionality, k, is the gradient of the graph.

Thus, whenever you get a straight-line graph passing through the origin, you know the corresponding equation will be of the form $y = kx$, and whenever you get an equation of the form $y = kx$ you know that the corresponding graph will be a straight line of gradient k passing through the origin. Furthermore, if the equation does not have the required form, then the graph will not be a straight line going through the origin, and if the graph is not a straight line going through the origin then the equation will not have the form $y = kx$. Remember though that in physical examples the x, y and k of the general equation may be replaced by much more complicated expressions. For example, page 114 of *Describing motion* tells us that the period, T, of a pendulum is related to its length, l, by the equation

DM page 114

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where g is the magnitude of the acceleration due to gravity (which we can assume to be constant). A graph of T against l would not be a straight-line graph passing through the origin, as the equation does not have the required form. However, a graph of T against $\sqrt{l}$ *would* be a straight line passing through the origin, as we would be plotting T along the vertical axis and $\sqrt{l}$ along the horizontal axis. The constant of proportionality, equal to the gradient of the graph, would be $\dfrac{2\pi}{\sqrt{g}}$. A graph of T^2 against l would also be a straight line passing through the origin: its gradient would be $\dfrac{4\pi^2}{g}$.

Quantities related in such a way that if one quantity halves, the other quantity doubles, are said to be *inversely proportional*. The pressure P and volume V of a fixed amount of gas at constant temperature are inversely proportional:

$$P \propto \frac{1}{V}, \text{ i.e. } P = \frac{k}{V} \text{ where } k \text{ is a constant.}$$

A graph of P against V is a curve (of a shape called a *hyperbola*, Figure 10a), but graphs of P against $1/V$ and of $1/P$ against V will both be straight lines through the origin (Figure 10b, c).

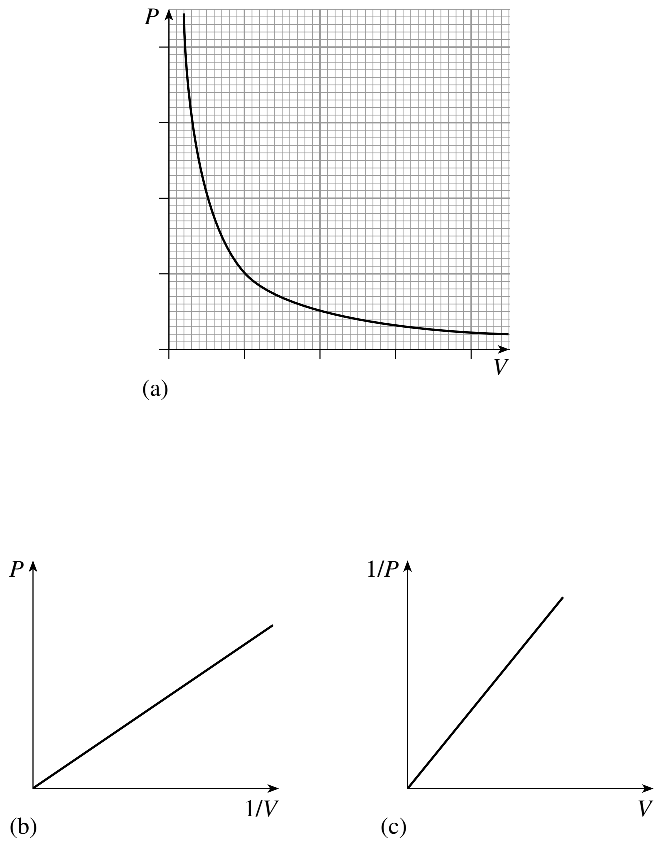

Figure 10 Graphs showing the way in which the pressure P of a fixed amount of gas at constant temperature depends on its volume V.

Question 21 Identify which of the following will be straight-line graphs passing through the origin; and for those that are, state the gradient:

(a) y against x when $y = 3x$,

(b) y against x when $y = 2x + 1$,

(c) p against v when $p = mv$,

(d) E_{trans} against v when $E_{\text{trans}} = \frac{1}{2}mv^2$ and m is constant,

(e) E_{trans} against v^2 when $E_{\text{trans}} = \frac{1}{2}mv^2$ and m is constant. ■

9.5 Straight-line graphs not passing through the origin

If we plot one quantity against another and get a straight line that crosses the axes at points away from the origin, then the quantities are not proportional to each other. However, since the graph is a straight line, we can still say that there is a *linear relation* between them.

Consider as an example the graph you plotted in Question 19 and whose gradient you found in Question 20. Recall that this graph does not go through the origin and that its gradient is 0.001 m^{-2}.

The points at which the line crosses the axes are called the *intercepts*. In this particular example, the intercept on the vertical axis is 0.2 m. This corresponds to the value of d when $V = 0$, i.e. it corresponds to the amount of water already in the pool.

The general equation for a straight-line graph of y versus x is

$$y = mx + c \tag{9.2}$$

where m is the gradient and c is the intercept on the vertical axis, i.e. the value of y when $x = 0$ and m and c are both constants. You may come across this equation written in different forms, e.g. $z = ky + c$ (where k and c are constants), $x = At + B$ (where A and B are constants), $y = a_0 + a_1x$ (where a_0 and a_1 are constants). It does not matter which form of the equation you remember and use.

Question 22 Calculate the gradient and hence write down the equation of the graphs shown in Figure 11. ■

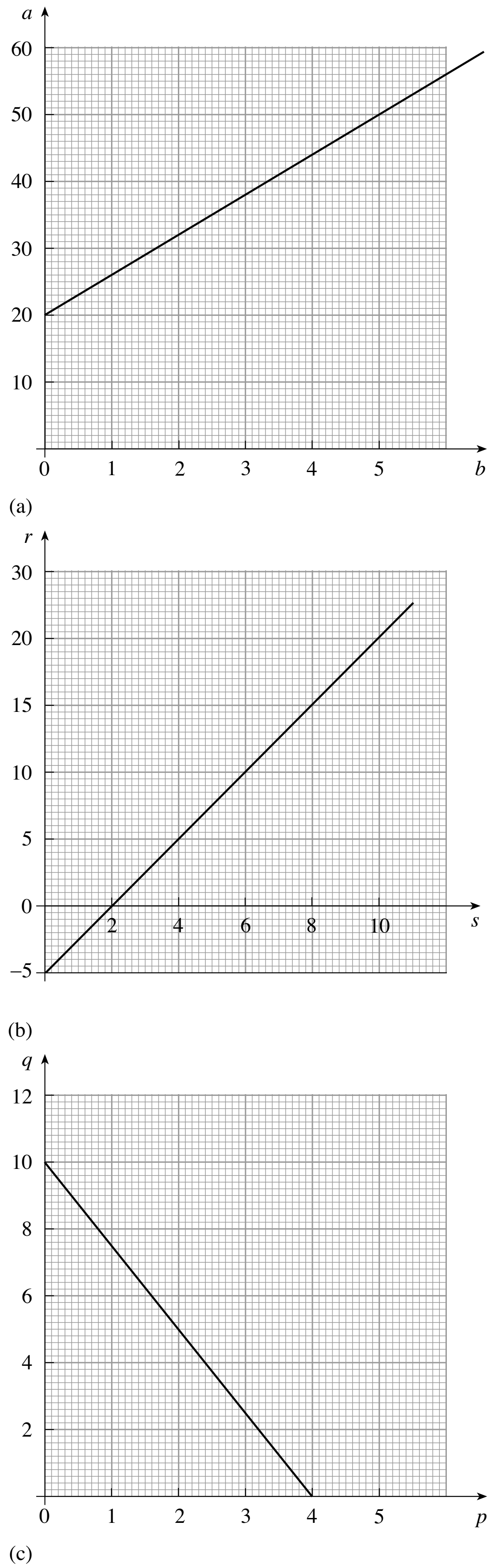

Figure 11 Graphs (a), (b) and (c) for use in Question 22.

Question 23 Sketch graphs to represent the following functions:

(a) $y = 3x + 1$, (b) $x = \frac{t}{3} - 1$, (c) $a = -2b + 3$. ■

10 Quadratic and other polynomial functions

You should now be reasonably familiar with linear functions of the form $f(t) = At + B$. Another simple class of functions consists of those of the form

$$f(t) = At^n$$

where A is a constant and n is a positive whole number such as 0, 1, 2, 3…. Functions of this kind include *squares*, such as $f(t) = 5t^2$ (in this case $A = 5$ and $n = 2$) and *cubes* such as $f(t) = 2t^3$ ($A = 2$ and $n = 3$). In general the graphs of functions of this type are not straight-line graphs, as Figure 12 shows, but the function that arises when $n = 0$ is an exception. In this case

$$f(t) = At^0 \qquad \text{(since } t^0 = 1\text{)},$$

i.e. the function is a constant and has the same value whatever the value of t.

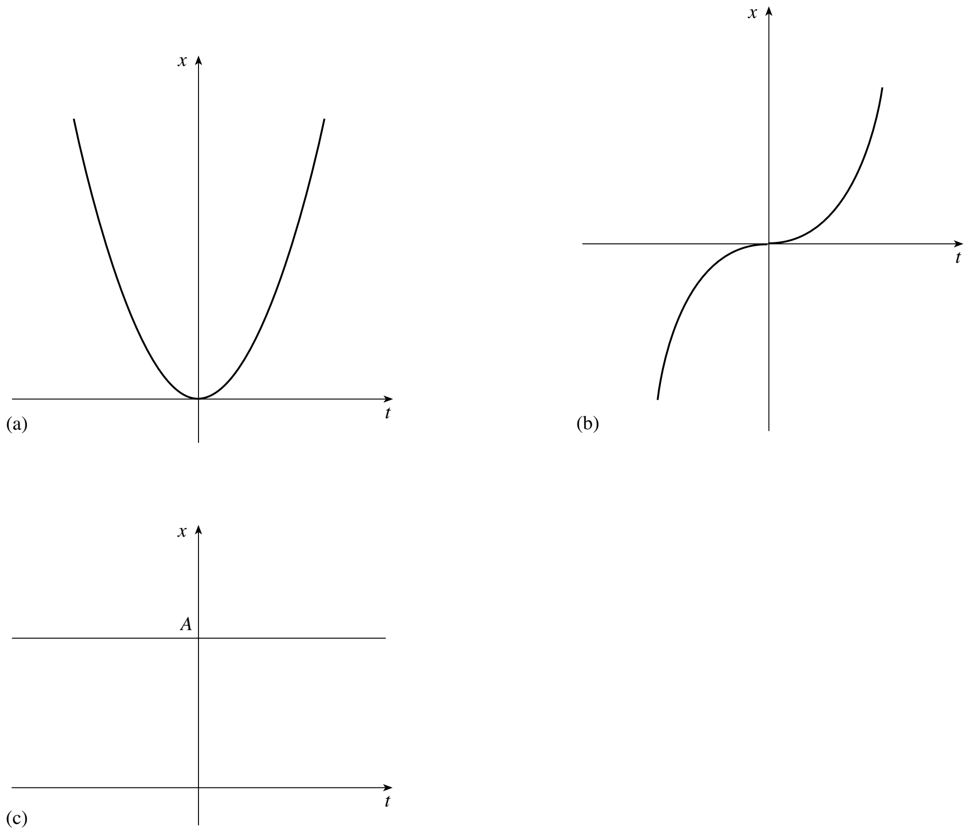

Figure 12 Sketch graphs of the functions (a) $x = f(t) = 5t^2$, (b) $x = f(t) = 2t^3$, (c) $x = f(t) = A$, where A is a constant.

When $n = 1$, the function is linear, and the graph is a straight line passing through the origin, as discussed in Section 9.4.

Polynomial functions are *sums* of squares, cubes, etc. Linear functions are a special case of polynomial function, as are the following (where A, B, C, D are constants):

$$f(t) = At^2 + Bt + C \qquad \text{(quadratic functions)}$$

$$f(t) = At^3 + Bt^2 + Ct + D \qquad \text{(cubic functions).}$$

Let's concentrate on quadratic functions, in particular two examples, $f(t) = 2t^2 + 5t + 3$ and $f(y) = -y^2 + 4y$. The graphs of these functions are shown in Figure 13.

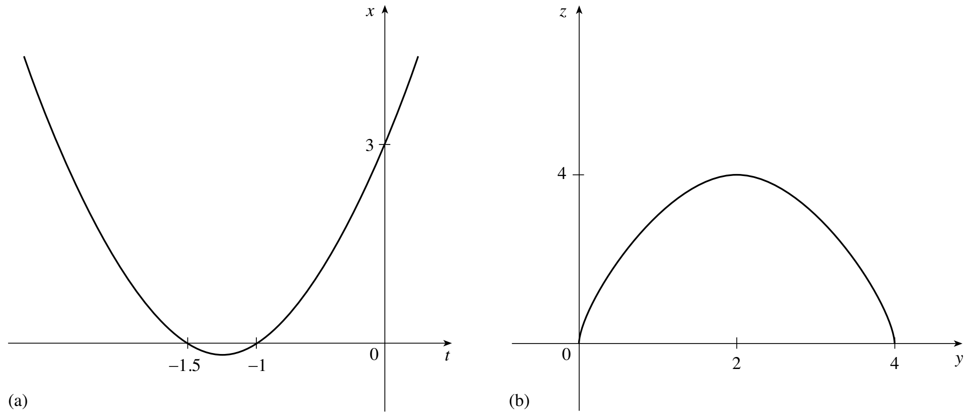

Figure 13 Sketch graphs of the functions (a) $x = f(t) = 2t^2 + 5t + 3$, (b) $z = f(y) = -y^2 + 4y$.

Figure 13a shows that $x = f(t) = 0$ occurs when the curve crosses the horizontal axis, which is at values of $t = -1.5$ and $t = -1$. These values of t are the *solutions of the quadratic equation* $2t^2 + 5t + 3 = 0$. Similarly (from Figure 13b), $z = f(y) = 0$ at $y = 0$ and $y = 4$, thus $y = 0$ and $y = 4$ are solutions of the quadratic equation $-y^2 + 4y = 0$.

It isn't always possible to obtain exact solutions of quadratic equations by a graphical method, and you may not want to plot a graph in any case, so two alternative methods are used to solve quadratic equations.

10.1 Solving quadratic equations by factorization

Recall from Section 1.2 that $a^2 - b^2 = (a + b)(a - b)$; we checked this result in Example 1.2, by multiplying out the brackets. In a similar way, some quadratic functions can be expressed as a product of terms, for example the function we considered in Figure 13b can be written

$$f(y) = -y^2 + 4y = y(-y + 4)$$

and the solutions of the quadratic equation $-y^2 + 4y = 0$ will occur when

$$-y^2 + 4y = y(-y + 4) = 0.$$

Recalling that multiplying by zero gives zero, this implies that either $y = 0$ or $-y + 4 = 0$ (i.e. $y = 4$), thus the solutions of the quadratic equation $-y^2 + 4y = 0$ are $y = 0$ and $y = 4$. Reassuringly, this is the same result as was implied by Figure 13b.

This method for solving quadratic equations by factorization can be applied more generally. Consider the general form of a quadratic equation

$$ax^2 + bx + c = 0 \tag{10.1}$$

where a, b and c are known constants and a is non-zero.

We need to start by rearranging Equation 10.1 in the form

$$a(x - \alpha)(x - \beta) = 0 \tag{10.2}$$

where α and β are constants that we will need to find (but don't worry about this yet). Then (since $a \neq 0$) the solutions of the equation will be $x = \alpha$ and $x = \beta$, since if $x = \alpha$ then $x - \alpha = 0$ so the left-hand side of Equation 10.2 will be zero, and similarly for $x = \beta$.

But how do we find α and β? This requires a certain amount of practice, but the following may help:

Multiplying out the brackets on the left-hand side of Equation 10.2 gives

$$a\{x(x - \beta) - \alpha(x - \beta)\} = 0$$

$$a\{x^2 - \beta x - \alpha x + \alpha\beta\} = 0.$$

Multiplying throughout by a gives

$$ax^2 - a\beta x - a\alpha x + a\alpha\beta = 0$$

and regrouping terms,

$$ax^2 + [-a(\alpha + \beta)]x + a\alpha\beta = 0.$$

Comparing this with Equation 10.1 shows that

$$-a(\alpha + \beta) = b \quad \text{and} \quad a\alpha\beta = c.$$

Thus we require that

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}.$$

In other words, in searching for solutions to the quadratic equation $ax^2 + bx + c = 0$ by factorization, we must look for solutions whose sum is $-\frac{b}{a}$ and whose product is $\frac{c}{a}$.

Example 10.1 Solve the equation $2t^2 + 5t + 3 = 0$ by factorization.

By comparison with Equation 10.1, we can see that in this case $a = 2$, $b = 5$ and $c = 3$, so we are looking for solutions whose sum is $-b/a = -5/2$ and whose product is $c/a = 3/2$. Two possible solutions which meet these criteria are $t = -1$ and $t = -1.5$, and we can check these by multiplying out:

$$\begin{aligned} 2(t+1)(t+1.5) &= 2\{t(t+1.5)+(t+1.5)\} \\ &= 2\{t^2+1.5t+t+1.5\} \\ &= 2t^2+5t+3. \end{aligned}$$

So $2t^2 + 5t + 3 = 0$ when $2(t + 1)(t + 1.5) = 0$, i.e. the solutions are $t = -1$ and $t = -1.5$. Again, this is the same result as was implied by Figure 13a. ■

Question 24 Solve the following equations (for s) by factorization:

(a) $4s^2 - 12s = 0$, (b) $s^2 - s - 6 = 0$. ■

10.2 Solving quadratic equations by formula

Using factorization to solve quadratic equations works well for simple equations with simple solutions, but it can be tedious and doesn't work if the solutions are not whole numbers or simple fractions. Fortunately help is at hand! It can be shown that *any* quadratic equation with the general form

$$ax^2 + bx + c = 0 \tag{10.1}$$

will have two solutions given by the *quadratic equation formula*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{10.3}$$

You do not need to remember this formula as it is given in the list of equations provided for the S207 examination.

If $b^2 > 4ac$ (i.e. b^2 is greater than $4ac$) then $b^2 - 4ac$ will be positive, and the formula will lead to two distinct solutions.

If $b^2 = 4ac$ then $b^2 - 4ac = 0$, so the two solutions will be identical ($x = -b/2a$).

If $b^2 < 4ac$ (i.e. b^2 is less than $4ac$) then $b^2 - 4ac$ will be negative and the solutions will involve the square root of a negative number. This situation, which corresponds to a graph which does not cross the horizontal axis, involves the use of *complex numbers*, and you will not be required to deal with such cases in S207.

A note on complex numbers

On a couple of occasions, the phrase 'complex number' is used in S207. An understanding of complex numbers is not required for the course, but the following is provided for background interest.

Complex numbers are generally written in the form $a + \mathrm{i}b$, where a and b are ordinary decimal numbers (called *real* numbers in this context).

The symbol 'i' is an algebraic symbol with the property $\mathrm{i}^2 = -1$. No real number can satisfy the equation $\mathrm{i}^2 = -1$, so i is said to be an *imaginary number* and is usually referred to as the 'square root of minus one'.

$a + \mathrm{i}b$ can be thought of as resembling a vector (See Section 16) with one real component and one imaginary component.

Example 10.2 Let's return to Example 10.1, and use the formula method to find the solutions of the equation $2t^2 + 5t + 3 = 0$.

Comparison with Equation 10.1 shows that $a = 2$, $b = 5$ and $c = 3$ on this occasion, so the solutions are

$$\begin{aligned} t &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-5 \pm \sqrt{5^2 - (4 \times 2 \times 3)}}{2 \times 2} \\ &= \frac{-5 \pm \sqrt{25 - 24}}{4} \\ &= \frac{-5 \pm 1}{4} \end{aligned}$$

So $t = \dfrac{-5+1}{4} = \dfrac{-4}{4} = -1$ or $t = \dfrac{-5-1}{4} = \dfrac{-6}{4} = -1.5$.

It is worth getting into the habit of checking the solutions you have found, by substituting back into the original equation:

For $t = -1$, $2t^2 + 5t + 3 = 2(-1)^2 + 5(-1) + 3 = 2 - 5 + 3 = 0$, as expected.

For $t = -1.5$, $2t^2 + 5t + 3 = 2(-1.5)^2 + 5(-1.5) + 3 = 4.5 - 7.5 + 3 = 0$, as expected.

So the solutions of the equation are $t = -1$ and $t = -1.5$. ■

Note that we have now obtained the same result by three different methods: graphically, by factorization, and using Equation 10.3. However, the formula method for solving quadratic equations comes into its own when you need to solve a quadratic equation which does not have solutions which are whole numbers or simple fractions, or when the constants a, b and c have not been assigned numerical values. Examples 10.3 and 10.4 consider such situations.

Example 10.3 Solve the quadratic equation $y^2 - 7y + 3 = 0$ for y.

Comparison with Equation 10.1 shows that $a = 1$, $b = -7$ and $c = 3$ on this occasion, so the solutions are

$$\begin{aligned} y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-7) \pm \sqrt{(-7)^2 - (4 \times 1 \times 3)}}{2 \times 1} \\ &= \frac{+7 \pm \sqrt{49 - 12}}{2} \\ &= \frac{7 \pm \sqrt{37}}{2}. \end{aligned}$$

So $y = \dfrac{7 + \sqrt{37}}{2} = 6.5414$ to 5 significant figures

or $y = \dfrac{7 - \sqrt{37}}{2} = 0.4586$ to 4 significant figures.

Note that it would not normally be sensible to quote answers to this large number of significant figures, but we need to keep a fairly large number of significant figures to enable us to check the solutions.

Checking for $y = 6.5414$ gives:

$$y^2 - 7y + 3 = (6.5414)^2 - 7(6.5414) + 3 \approx 0.0001 \approx 0$$

and for $y = 0.4586$ gives:

$$y^2 - 7y + 3 = (0.4586)^2 - 7(0.4586) + 3 \approx 0.0001 \approx 0.$$

So both solutions are correct to within the accuracy to which we have evaluated $\sqrt{37}$. ■

Example 10.4

On page 75 of *Describing motion* you are required to solve the following equation for an unknown t: *DM* page 75

$$\tfrac{1}{2}gt^2 - u_y t - h = 0.$$

Values for g, u_y and h are not given, but you know they are all positive constants. Comparison with Equation 10.1 gives

$$a = \frac{1}{2}g = \frac{g}{2}\text{, } b = -u_y \text{ and } c = -h.$$

Then the formula gives the solutions for t as

$$\begin{aligned} t &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-u_y) \pm \sqrt{(-u_y)^2 - \{4 \times \frac{g}{2} \times (-h)\}}}{2 \times \frac{g}{2}} \\ &= \frac{u_y \pm \sqrt{u_y^2 + 2gh}}{g}. \end{aligned}$$

Since g and h are positive constants, the quantity $\sqrt{u_y^2 + 2gh}$ will be bigger than u_y, so $\dfrac{u_y - \sqrt{u_y^2 + 2gh}}{g}$ will be *negative*. The physics of the situation under consideration (t is a time of flight) means that this is not a physically acceptable solution, so the only solution is

$$t = \frac{u_y + \sqrt{u_y^2 + 2gh}}{g}. \quad ■$$

Question 25 Use the Equation 10.1 to solve the following quadratic equations then check your answers.

(a) $s^2 - s - 6 = 0$, (b) $2y^2 + 5y + 1 = 0$. ■

11 Differentiation

When a graph is drawn of x against t, say, the rate of change of x in response to changes in t is simply the gradient of the graph. For quantities following a linear relation, the gradient will be a constant which can be calculated easily from the graph, but it isn't quite so simple to find the gradient when the graph is not a straight line. Fortunately, if the algebraic form of the function is known, then we can calculate the rate of change of x with respect to t by a technique known as *differentiation*. We will not attempt to describe in great detail the principles that underpin differentiation, but we will introduce some of the general ideas, list some results, and show you how to use them.

11.1 Curved graphs

The *tangent* at a point on a curve is a straight line whose slope matches the tilt of the curve at that point. The tangent touches the curve at that point, but does not cut across the curve.

The gradient at any point on a curved graph is simply the gradient of the tangent to the curve at that point. In Figure 14, the gradient is very small for small values of t, becomes much higher for intermediate values of t, and falls to small values again for large values of t. So, whereas the gradient of a straight-line graph was the same for all values of t, for a curved graph the gradient changes depending on which part of the curve is considered.

It would be impractical to draw a tangent for every point on a curve to calculate the gradient, but some simple mathematical tools come to our aid. With reference to Figure 14, the gradient of the tangent at some point P can be approximated by the gradient of a straight line joining two points on either side of P, shown in the figure as $\frac{\Delta x}{\Delta t}$. This approximation is rather coarse if the chosen side points are too far away from the central point of interest P, but it becomes more accurate the closer the side points come to P.

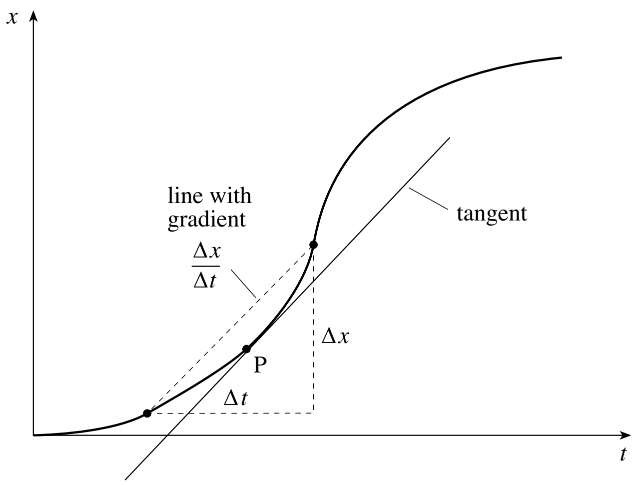

Figure 14 The gradient at a point P on a curved line is equal to the gradient of the tangent at that point.

In the limit where the points are so close to P as to be indistinguishable, the intervals Δx and Δt become infinitesimal, and the approximation becomes *exact*. The gradient in this limit is written mathematically not as $\frac{\Delta x}{\Delta t}$ but as $\frac{\mathrm{d}x}{\mathrm{d}t}$, the lowercase 'd's signifying the limit of infinitesimal intervals. That is, the gradient of a graph of x versus t at some point is described as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}. \tag{11.1}$$

Thus $\frac{\mathrm{d}x}{\mathrm{d}t}$ represents the rate of change of x with t, and is known mathematically as 'the *derivative* of x with respect to t'. The value of $\frac{\mathrm{d}x}{\mathrm{d}t}$ usually changes from one point to the next, unless x is a linear function of t. Generally, if x is a function of t, then the derivative $\frac{\mathrm{d}x}{\mathrm{d}t}$ will also be a function of t.

When $\frac{\mathrm{d}x}{\mathrm{d}t}$ (the gradient of a graph of x versus t) changes with t, it is often useful to ask *how rapidly* the *gradient* changes with t. We can determine this by applying the same process described above, not to x but to $\frac{\mathrm{d}x}{\mathrm{d}t}$, to find the gradient *of* $\frac{\mathrm{d}x}{\mathrm{d}t}$ versus t. We would then be finding the derivative of the first derivative. Such a quantity is referred to

as 'the *second derivative* of x with respect to t', and extending the notation above, it is written $\dfrac{d^2x}{dt^2}$.

It would be wrong to think of $\dfrac{dx}{dt}$ as a ratio of the quantities dx and dt. Instead, it is useful to regard $\dfrac{dx}{dt}$ as consisting of an entity $\dfrac{d}{dt}$ (called a mathematical *operator*) that acts on the function $x(t)$ which follows. Similarly $\dfrac{d^2x}{dt^2}$ can be regarded as the operator $\dfrac{d^2}{dt^2}$ acting on $x(t)$. Note that the second derivative is written $\dfrac{d^2x}{dt^2}$ *not* $\dfrac{dx^2}{dt^2}$. Remembering that it is the operator $\dfrac{d}{dt}$ that is to be squared rather than the function $x(t)$ may help you to avoid this error.

Some books use shorthand notations for first and second derivatives. Although we do not use them in *The Physical World*, you should be aware of them in case you encounter them in other texts. Sometimes $\dfrac{dx}{dt}$ and $\dfrac{d^2x}{dt^2}$ are written as x' and x'' (called 'x prime' and 'x double prime'). This notation is used particularly in conjunction with function notation, so $f'(y)$ (or just f') means $\dfrac{df(y)}{dy}$ and $f''(y)$ (or just f'') means $\dfrac{d^2f(y)}{dy^2}$. Note that primes are also used for completely different topics having nothing to do with derivatives; the context of the problem will usually prevent confusion.

11.2 Differentiation of known functions

If we know the *analytic* or *functional form* (the equation) of the relationship between two variables, then the derivative can be obtained without having to draw a graph. As an example let's return to the function used extensively in Section 10, $x = f(t) = 2t^2 + 5t + 3$. The derivative of x with respect to t, hence the gradient of the graph shown in Figure 13a is given by

$$\begin{aligned}\frac{dx}{dt} &= \frac{df(t)}{dt}\\ &= \frac{d}{dt}(2t^2+5t+3)\\ &= 4t+5.\end{aligned}$$

It may appear that we have pulled this result out of a hat, but we have simply applied a set of standard rules, which are reproduced in Table 3.

Table 3 Some functions and their derivatives: A and n are constants, and x, y and z are functions of t.

Function, x	Derivative, $\dfrac{dx}{dt}$
A	0
t^n	nt^{n-1}
Ay	$A\dfrac{dy}{dt}$
$y + z$	$\dfrac{dy}{dt}+\dfrac{dz}{dt}$

It is worth emphasizing two points about the results given in Table 3:

1 Although we have not attempted to prove the results, it is quite straightforward to do so (though not required for S207), simply by applying the definition of a derivative, $\dfrac{dx}{dt} = \lim\limits_{\Delta t\to 0}\dfrac{\Delta x}{\Delta t}$, given in Section 11.1. In each case the derivative gives the gradient of the graph of the function.

2 You do not need to learn the results given in Table 3, and for convenience we have given the results in exactly the same form as they are given in the list of standard equations and constants provided for the S207 examination. (Table 3 will be extended slightly as we introduce three more functions later in this handbook.) However, you *do* need to know how to use the results. To help you with this we will consider each entry in Table 3 in some detail.

Table 3 tells us that

$$\text{if } x = A, \text{ then } \frac{\mathrm{d}x}{\mathrm{d}t} = 0. \qquad (11.2)$$

In other words, differentiating a constant always gives zero. This is consistent with the assertion that the derivative is equivalent to the gradient of the graph, as the gradient of a graph of a horizontal line will always be zero (see Figure 15).

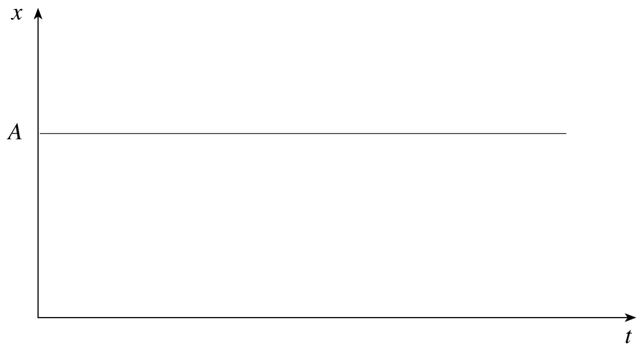

Figure 15 The graph of a constant function $x = A$. Note that its gradient is zero.

Table 3 also tells us that

$$\text{if } x = t^n \text{ then } \frac{\mathrm{d}x}{\mathrm{d}t} = nt^{n-1}. \qquad (11.3)$$

What does this mean? Some examples should make things clearer:

Example 11.1 If $x = t^3$, what is $\frac{\mathrm{d}x}{\mathrm{d}t}$?

In this case $n = 3$, so

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2.$$

Note that in the derivative t is raised to *one power less* than in the original function, and the result is *multiplied* by the original power (three, in this case). ■

Example 11.2 If $z = y^5$, what is $\frac{\mathrm{d}z}{\mathrm{d}y}$?

In this case the variables are z and y instead of x and t, but the principles are the same. Since $n = 5$

$$\frac{\mathrm{d}z}{\mathrm{d}y} = 5y^4. \quad ■$$

Equation 11.3 also applies when n is negative or fractional, as illustrated in Examples 11.3 and 11.4.

Example 11.3 If $y = \frac{1}{x^2}$, what is $\frac{\mathrm{d}y}{\mathrm{d}x}$?

Remember, from Section 3 of this handbook, that $\frac{1}{x^2}$ can be written as x^{-2}, so

$y = \frac{1}{x^2} = x^{-2}$, i.e. $n = -2$. Thus

$$\frac{dy}{dx} = -2x^{-3} = \frac{-2}{x^3}.$$

Note that, once again, when differentiating, the power was reduced by one (from –2 to –3) and the result was multiplied by the original power (–2). ■

Example 11.4 If $b = \sqrt{a}$, what is $\dfrac{\mathrm{d}b}{\mathrm{d}a}$?

Remember, from Section 3 of this handbook, that $\sqrt{a}$ can be written as $a^{1/2}$, so, $b = \sqrt{a} = a^{1/2}$, i.e. $n = \frac{1}{2}$. Thus

$$\frac{\mathrm{d}b}{\mathrm{d}a} = \tfrac{1}{2}a^{-1/2} = \frac{1}{2} \times \frac{1}{a^{1/2}} = \frac{1}{2\sqrt{a}}.$$

Again, in differentiating, the power was reduced by one (from $\frac{1}{2}$ to $-\frac{1}{2}$) and the result was multiplied by the original power ($\frac{1}{2}$). ■

Returning to Table 3 we see that

$$\text{if } x = Ay \text{ then } \frac{\mathrm{d}x}{\mathrm{d}t} = A\frac{\mathrm{d}y}{\mathrm{d}t} \tag{11.4}$$

where x and y are functions of t and A is a constant. In other words, differentiating the product of a constant and a function results in the product of the same constant and the derivative of the function, as Example 11.5 illustrates.

Example 11.5 If $x = 6t^3$, what is $\dfrac{\mathrm{d}x}{\mathrm{d}t}$?

Comparison with Equation 11.4 shows that $y = t^3$ and $A = 6$ on this occasion. We differentiated t^3 in Example 11.1, so we know that for $y = t^3$, $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 3t^2$. So

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A\frac{\mathrm{d}y}{\mathrm{d}t} = 6 \times 3t^2 = 18t^2. \quad ■$$

The final rule in Table 3 is that

$$\text{if } x = y + z \text{ then } \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\mathrm{d}z}{\mathrm{d}t} \tag{11.5}$$

where x, y and z are all functions of t. This means that differentiating the sum of two functions results in the sum of the derivatives of the individual functions, as is shown in Example 11.6.

Example 11.6 If $x = t^5 + 6t^3$, what is $\dfrac{\mathrm{d}x}{\mathrm{d}t}$?

Setting $y = t^5$ and $z = 6t^3$, we already know (from Examples 11.2 and 11.5) that $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 5t^4$ and $\dfrac{\mathrm{d}z}{\mathrm{d}t} = 18t^2$, so

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\mathrm{d}z}{\mathrm{d}t} = 5t^4 + 18t^2. \quad ■$$

In the preceding examples, we started by using the rules from Table 3 in isolation, but in Example 11.6 we have combined three of the rules (Equations 11.3, 11.4 and 11.5) in order to arrive at the final result. Similarly, in the example given right at the beginning of this section, we used *all* of the rules from Table 3 to show that the derivative of $x = 2t^2 + 5t + 3$ is

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(2t^2 + 5t + 3)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}(2t^2) + \frac{\mathrm{d}}{\mathrm{d}t}(5t) + \frac{\mathrm{d}}{\mathrm{d}t}(3)\\ &= 4t + 5.\end{aligned}$$

Applying the rules will become straightforward with practice, and there are some examples for you to try for yourself at the end of this section. Before trying these examples, have a look at three more worked examples, which are intended to illustrate

how you can apply the simple rules of differentiation to the physics you will meet in S207.

DM page 18

Example 11.7 Differentiate $x(t) = \upsilon_x t + x_0$ (from *Describing motion* page 18), where υ_x and x_0 are constants, to give the first derivative of x with respect to t. Differentiate again, to give the second derivative of x with respect to t.

$x(t)$ is the sum of two functions, so its derivative is the sum of the derivatives of the individual functions, that is,

$$\frac{dx}{dt} = \frac{d}{dt}(\upsilon_x t + x_0) = \frac{d}{dt}(\upsilon_x t) + \frac{d}{dt}(x_0)$$

$$\frac{d}{dt}(\upsilon_x t) = \upsilon_x \times 1 \times t^0 = \upsilon_x,$$

since $n = 1$ and $t^0 = 1$ on this occasion, and x_0 is a constant, so $\frac{d}{dt}(x_0) = 0$. So

$$\frac{dx}{dt} = \upsilon_x + 0 = \upsilon_x.$$

Note that the derivative of x with respect to t is a constant. This should not surprise you, since the graph of the function $x(t) = \upsilon_x t + x_0$ is a straight line, as shown in Figure 16a.

The second derivative of x with respect to t is zero, since differentiating a constant always gives zero:

$$\frac{d^2 x}{dt^2} = \frac{d}{dt}(\upsilon_x) = 0. \quad \blacksquare$$

DM page 37

Example 11.8 Differentiate $x(t) = x_0 + u_x t + \frac{1}{2} a_x t^2$ (from *Describing motion* page 37), where x_0, u_x and a_x are constants, to give the first derivative of x with respect to t. Differentiate again, to give the second derivative of x with respect to t.

We now have the sum of three functions, so the derivative is

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}(x_0 + u_x t + \tfrac{1}{2} a_x t^2) \\ &= \frac{d}{dt}(x_0) + \frac{d}{dt}(u_x t) + \frac{d}{dt}(\tfrac{1}{2} a_x t^2)\end{aligned}$$

Now

$$\frac{d}{dt}(x_0) = 0, \; \frac{d}{dt}(u_x t) = u_x \times 1 \times t^0 = u_x, \text{ and}$$

$$\frac{d}{dt}(\tfrac{1}{2} a_x t^2) = \tfrac{1}{2} a_x \times 2t^1 = a_x t,$$

so

$$\frac{dx}{dt} = 0 + u_x + a_x t = u_x + a_x t.$$

Differentiating again gives

$$\begin{aligned}\frac{d^2 x}{dt^2} &= \frac{d}{dt}(u_x + a_x t) \\ &= \frac{d}{dt}(u_x) + \frac{d}{dt}(a_x t) \\ &= 0 + a_x \\ &= a_x. \quad \blacksquare\end{aligned}$$

Graphs of $x(t)$, $\frac{dx}{dt}$ and $\frac{d^2 x}{dt^2}$ for the functions given in Examples 11.7 and 11.8 are shown in Figure 16.

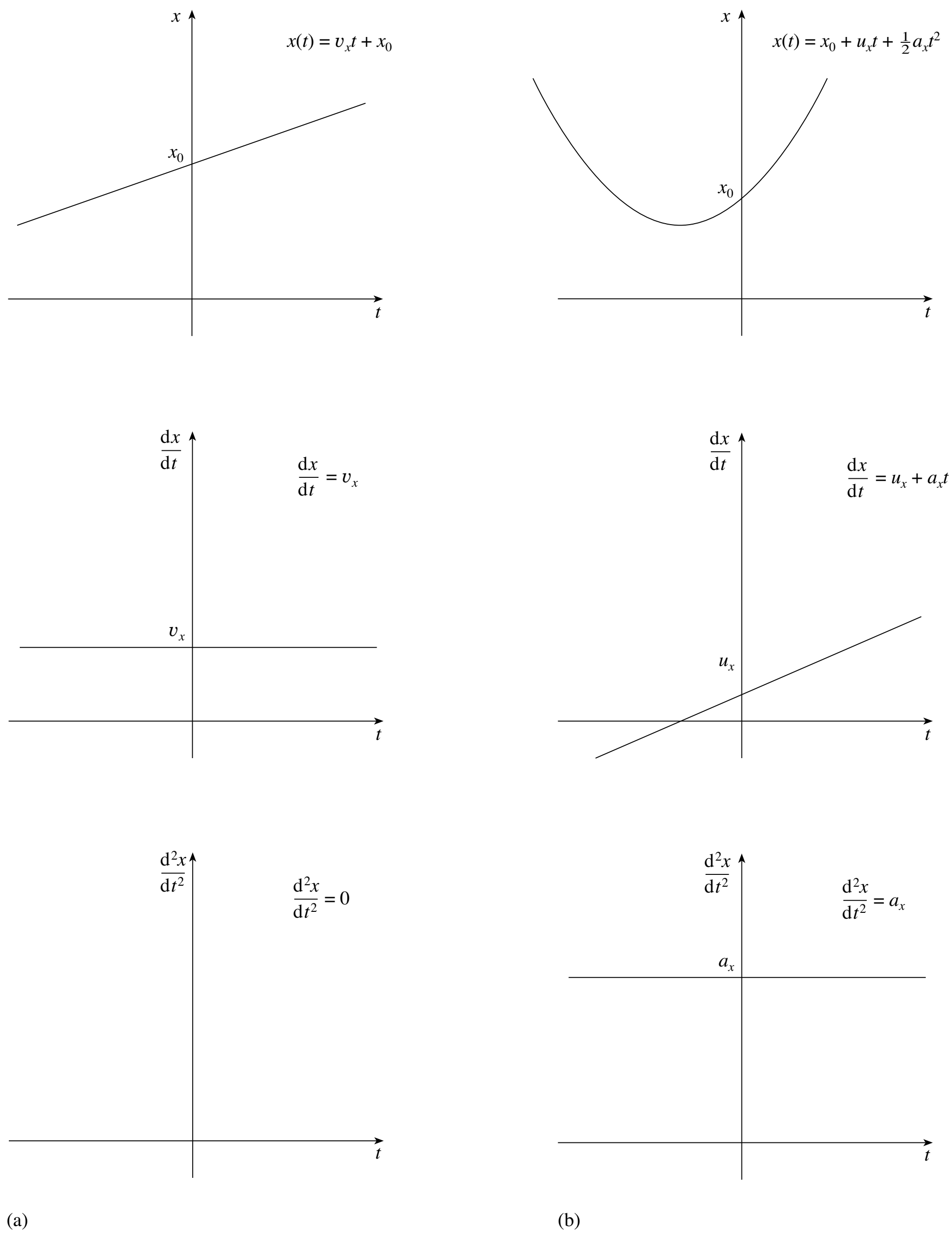

Figure 16 (a) The function $x(t) = v_x t + x_0$ and its first and second derivatives; (b) the function $x(t) = x_0 + u_x t + \frac{1}{2}a_x t^2$ and its first and second derivatives.

Example 11.9 Differentiate $E(r) = -\dfrac{GmM}{r}$ where G, m and M are constants, to give the first derivative of E with respect to r.

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Note that this example is very similar to Question 2.13 on page 90 of *Predicting motion*, but we will perform the differentiation without having to use the hint.

We can write the function as $E(r) = -GmMr^{-1}$, so

$$\begin{aligned}\frac{dE}{dr} &= \frac{d}{dr}(-GmMr^{-1})\\ &= -GmM(-1)r^{-2}\\ &= \frac{GmM}{r^2}. \ \blacksquare\end{aligned}$$

Question 26 Differentiate the following functions

(a) $x(t) = t^7$, (b) $y(x) = \dfrac{1}{x^3}$

(c) $w(t) = 3\sqrt{t}$, (d) $z(y) = 4y^2 + y$. ■

Question 27 Find the first and second derivatives of x with respect to t for the function $x(t) = At^3 + Bt^2 + Ct + D$, where A, B, C and D are constants. ■

12 The exponential function

The *exponential function* is defined by the relation $\exp(x) = \mathrm{e}^x$, where e is the mathematical constant with the value e = 2.718 (to four significant figures) that was introduced in the discussion of natural logarithms in Section 5. The exponential function is important because

$$\text{if } x = \mathrm{e}^{kt}\text{, then } \frac{\mathrm{d}x}{\mathrm{d}t} = k\mathrm{e}^{kt} .$$

Since $k\mathrm{e}^{kt}$ is just kx, we can write

$$\text{if } x = \mathrm{e}^{kt}\text{, then } \frac{\mathrm{d}x}{\mathrm{d}t} = kx . \tag{12.1}$$

In other words, if $x = \mathrm{e}^{kt}$, then the derivative of x with respect to t (i.e. the gradient of the graph of x against t) is directly proportional to x itself. This is the special property of the number e that was hinted at in the earlier discussion of natural logarithms.

PM page 112

A process in which a quantity Z varies according to the relation $\dfrac{\mathrm{d}Z}{\mathrm{d}t} = kZ$ is referred to as an *exponential process* (see *Predicting motion* page 112) and may be described by a function of the form $Z(t) = Z_0\mathrm{e}^{kt}$, where Z_0 and k are constants. Here, Z_0 represents the initial value of Z, i.e. the value of Z when $t = 0$. Some exponential curves, corresponding to various values of Z_0 and k are shown in Figure 17.

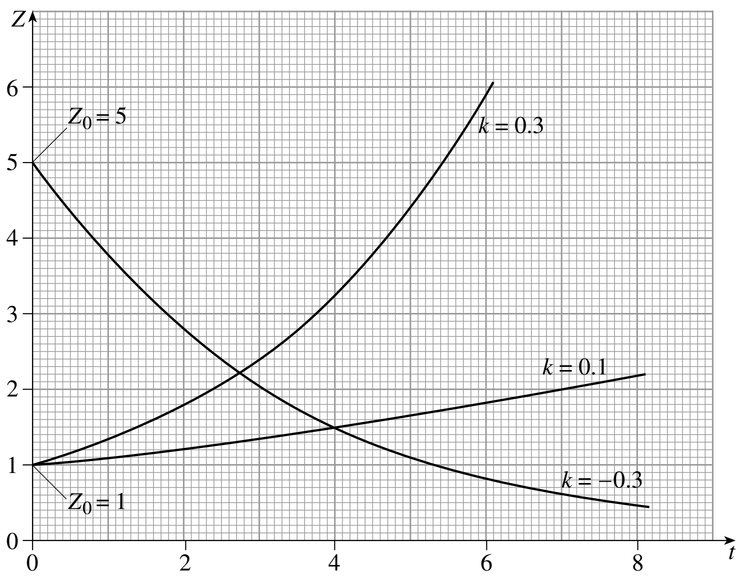


Figure 17 Exponential curves, $Z(t) = Z_0\mathrm{e}^{kt}$, for various values of Z_0 and k.

If k is positive, the process is one of *exponential growth* and may be described by $Z(t) = Z_0\mathrm{e}^{t/\tau}$ where $\tau = 1/k$. If k is negative, the process is one of *exponential decay* and may be described by $\tau = -1/k$. In either case, the positive constant τ is called the *time constant* of the exponential process and represents the time required by the varying quantity Z to increase or decrease by a factor of e.

An exponential decay curve describing the decay of 1.0×10^6 radioactive carbon nuclei is shown in Figure 18. In this particular example, if the number of undecayed nuclei

remaining in the sample at time t is represented by $N(t)$, then the decay process is described by

$$N(t) = N_0\,e^{-t/\tau} \text{ and } \frac{dN}{dt} = \frac{-N}{\tau}$$

where the time constant τ has the value 8100 years.

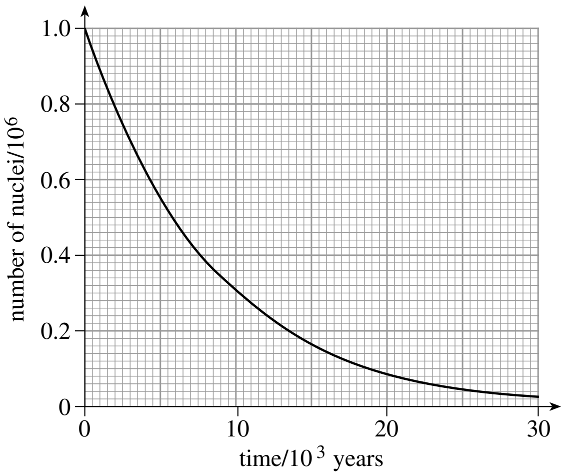


Figure 18 Example of an exponential decay, showing the number of radioactive carbon nuclei in a sample plotted against time. The time constant of this process is 8100 years (2.56×10^{11} s), and the initial number of nuclei is $N_0 = 1.0 \times 10^6$.

Question 28 Show, by differentiating, that if $N(t) = N_0 e^{-t/\tau}$ then $\dfrac{dN}{dt} = \dfrac{-N}{\tau}$. If $\tau =$ 8100 years and $N_0 = 1.0 \times 10^6$, find N at $t = 20\,000$ years. ■

Note that any function of the form $y = a^x$, where a is a positive constant, may be written as $y = e^{kx}$ since it is always possible to find a constant k such that $a = e^k$, and we can then write $a^x = (e^k)^x = e^{kx}$. A consequence of this is that if a particular decay process can be described by the relation $N(t) = N_0 e^{-t/\tau}$, then that process may also be described by

$$N(t) = N_0 2^{-t/\tau_{1/2}}\,,$$

where $\tau_{1/2} = \tau \log_e 2$ is called the *half-life* of the process and represents the time required for the decaying quantity to decrease by a factor of 2.

Example 12.1 Show that the time constant τ of an exponential decay process is related to the half-life of that process by $\tau_{1/2} = \tau \log_e 2$ as claimed above.

We can equate the two expressions for $N(t)$ to give

$$N(t) = N_0 e^{-t/\tau} = N_0 2^{-t/\tau_{1/2}}\,.$$

Dividing by N_0 gives

$$e^{-t/\tau} = 2^{-t/\tau_{1/2}}$$

and taking natural logarithms of both sides

$$\log_e e^{-t/\tau} = \log_e 2^{-t/\tau_{1/2}}.$$

In general, $\log_e a^b = b \log_e a$ (Equation 5.5), so in this case

$$\frac{-t}{\tau} \log_e e = \frac{-t}{\tau_{1/2}} \log_e 2\,.$$

Now $e^1 = e$, so $\log_e e = 1$, i.e.

$$\frac{-t}{\tau} = \frac{-t}{\tau_{1/2}} \log_e 2.$$

Multiplying both sides by $\tau_{1/2}$ and τ, and dividing throughout by $-t$ gives

$$\tau_{1/2} = \tau \log_e 2 \text{ as claimed. } ■$$

Question 29 Find (from the graph in Figure 18) an approximate value of the half-life of the exponential decay process shown, and use this to verify that $\tau_{1/2} = \tau \log_e 2$ is approximately correct. ■

13 Trigonometric functions

13.1 Sine and cosine graphs

In Section 8 we discussed the trigonometric ratios (sin, cos, tan) in the context of acute angles (0 rad $\leq \theta < \pi/2$ rad) within right-angled triangles, but angles in nature are not so constrained and may take any value $-\infty < \theta < +\infty$. (For example, if you turn around twice, you have turned through an angle of 720° = 4π radians.) Trigonometric ratios may be generalized to provide a set of trigonometric *functions* that are defined over the full range of angles, through the concept of a *unit circle* (i.e. a circle of radius 1, as shown in Figure 19).

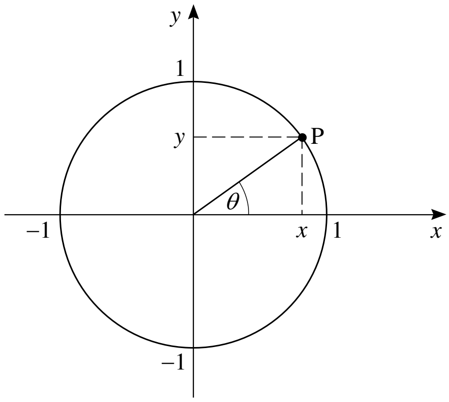


Figure 19 A unit circle.

As the radius arm of a unit circle sweeps out an angle from $\theta = 0$ (corresponding to the positive x-axis) to $\theta = \pi/2$ (corresponding to the positive y-axis), the x-coordinate of the tip of the radius arm indicates the values of cos θ and the y-coordinate indicates the value of sin θ. As θ increases beyond π/2, the x- and y-coordinates of the tip of the radius arm continue to indicate well-defined values and these may be used to define the trigonometric functions, cos(θ) and sin(θ). The x-coordinate, and hence cos(θ), becomes negative as θ exceeds π/2. The y-coordinate, and hence sin(θ), becomes negative as θ exceeds π. As the radius arm sweeps out the full range of angles from $\theta = 0$ to 2π, the full range of the trigonometric functions is revealed. For angles greater than 2π, the functions repeat themselves. Because $\sin(\theta + 2\pi) = \sin(\theta)$ and $\cos(\theta + 2\pi) = \cos(\theta)$, these two trigonometric functions are said to be *periodic* and to have *period* 2π, as shown in Figure 20.

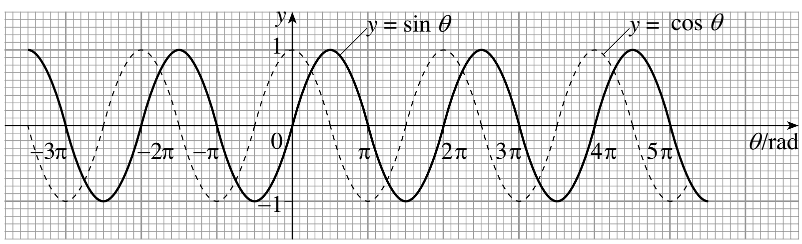


Figure 20 Graphs of sin θ and cos θ against θ.

The *argument* of a sine (or cosine) function is the expression or number whose sine (or cosine) is being computed. So the argument of sin(x) is x, the argument of sin(0.4) is 0.4, and so on. The argument of a trigonometric function is best thought of as a dimensionless (and unitless) number, that may have nothing to do with angles at all. However, it is often treated as an angle in radians or it may even be expressed as the equivalent number of degrees. Thus, for example, sin(180°) = sin(π) = 0 and cos(180°) = cos(π) = −1. Note that the brackets that normally enclose the argument of a function are, in practice, often omitted in the case of trigonometric functions.

The following relationships hold between the trigonometric functions. They are valid for arguments in both degrees and radians.

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\sin(90° \pm \theta) = \sin(\pi/2 \pm \theta) = \cos\theta$$

$$\cos(90° \pm \theta) = \cos(\pi/2 \pm \theta) = \mp\sin\theta$$

$$\sin(180° \pm \theta) = \sin(\pi \pm \theta) = \mp\sin\theta$$

$$\cos(180° \pm \theta) = \cos(\pi \pm \theta) = -\cos\theta$$

$$\sin(360° \pm \theta) = \sin(2\pi \pm \theta) = \pm\sin\theta$$

$$\cos(360° \pm \theta) = \cos(2\pi \pm \theta) = \cos\theta$$

When the symbol '±' precedes the symbol '∓', instances of '+' in the former lead to a '–' in the latter, and instances of a '–' in the former induce a '+' in the latter.

The following trigonometric functions may also be useful:

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin \mathrm{B}$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$2\sin A\cos B = \sin(A + B) + \sin(A - B)$$

$$2\cos A\cos B = \cos(A + B) + \cos(A - B)$$

$$2\sin A\sin B = \cos(A - B) - \cos(A + B)$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right).$$

If we plot a graph of the way one quantity varies with another and get either a *sine curve* or a *cosine curve*, we say in either case that the quantity plotted on the vertical axis varies *sinusoidally*.

An example of a quantity which varies sinusoidally with time is the voltage V of mains alternating current (AC) electricity. The variation of V with t is given by the equation $V = V_{\max}\sin(2\pi ft)$, as illustrated in Figure 21. $V_{\max}$ is known as the *amplitude*; this is the quantity that scales the sine curve.

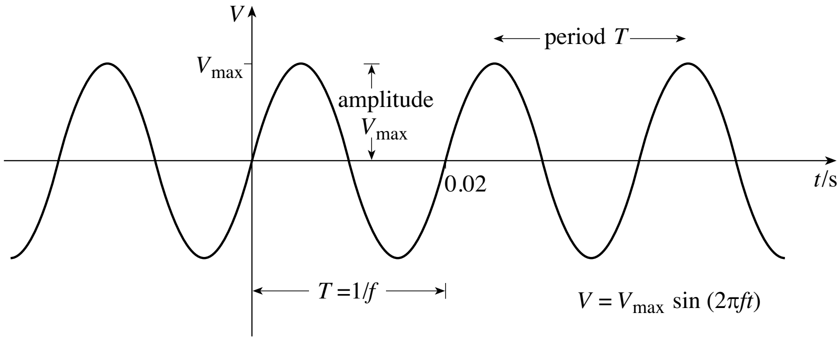


Figure 21 Mains voltage varies sinusoidally with time.

The argument $2\pi ft$ of the sine function includes the *frequency* f. (In the UK, the frequency of the mains is 50 Hz.) The greater the frequency, the shorter is the period of time in which the curve repeats itself.

Figure 20 shows that a sinusoidally varying quantity repeats exactly the same pattern of variation every time the argument of the sine function increases by 2π. In Figure 21, the argument $2\pi ft$ increases by 2π when ft increases by an increment of 1, i.e. when t increases by $1/f$. This time, $1/f$, is called the *period* of oscillation, T. Thus in Figure 21 the graph crosses the axis going positive at $t = 0$ (since $\sin(0) = 0$), and again, one full cycle later, at $t = 1/f$ where again $\sin(2\pi f \times 1/f) = \sin 2\pi = 0$.

Alternative expressions for a sinusoidal oscillation in time are thus:

$$y = A\sin(2\pi ft) \qquad \text{or} \qquad y = A\sin\left(\frac{2\pi t}{T}\right)$$

It is also useful to define an *angular frequency* $\omega = 2\pi f$, from which it follows that $y = A\sin(\omega t)$.

If an oscillation doesn't start with $y = 0$ at $t = 0$, the graphical representation of Figure 20 must be modified by displacing the sine curve (see Figure 22), and the algebraic representation supplemented by adding a constant term, known as the *initial phase* or *phase constant*, ϕ, to the argument, i.e.

$$y = A\sin\left(\frac{2\pi t}{T} + \phi\right) \qquad \text{or} \qquad y = A\sin(\omega t + \phi).$$

Note that the phase difference between a sine function and a cosine function is $\pi/2$, i.e. $\cos\theta = \sin(\theta + \pi/2)$.

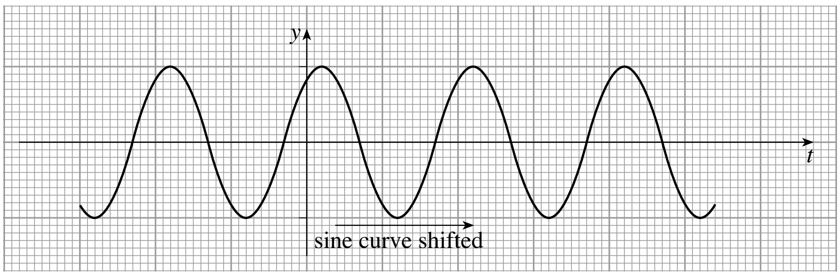


Figure 22 When the oscillation doesn't start with $y = 0$ at $t = 0$, the sine curve is shifted.

13.2 Differentiating sines and cosines

The standard derivatives of the sine and cosine functions are as follows:

$$\text{if } x = \sin(\omega t + \phi) \text{ then } \frac{dx}{dt} = \omega\cos(\omega t + \phi) \qquad (13.1)$$

$$\text{if } x = \cos(\omega t + \phi) \text{ then } \frac{dx}{dt} = -\omega\sin(\omega t + \phi) \qquad (13.2)$$

where ω and ϕ are constants and x is a function of t. As in Section 11, we will show you how to use these derivatives by worked examples, then provide some questions for you to try for yourself.

The derivatives of the exponential, sine and cosine functions are listed, along with those from Table 3, in the reference section at the end of this *Maths Handbook*.

Example 13.1 Differentiate $x(t) = 5\cos(2t)$.

The function is the product of a constant (5) and the function $x(t) = \cos(2t)$, so

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= 5\times\frac{\mathrm{d}}{\mathrm{d}t}\{\cos(2t)\}\\ &= 5\times\{-2\sin(2t)\}\\ &= -10\sin(2t). \quad \blacksquare\end{aligned}$$

Example 13.2 On page 111 of *Describing motion*, you are asked to find the first and second derivatives of x with respect to t for

DM page 111

$$x(t) = A\sin(\omega t + \phi)$$

where ω and ϕ are constants.

Again, this is a product of a constant and a function, so

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= A\times\frac{\mathrm{d}}{\mathrm{d}t}\{\sin(\omega t + \phi)\}\\ &= A\times\{\omega\cos(\omega t + \phi)\}\\ &= A\omega\cos(\omega t + \phi).\end{aligned}$$

Differentiating again gives

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= A\omega\times\frac{\mathrm{d}}{\mathrm{d}t}\{\cos(\omega t + \phi)\}\\ &= A\omega\times\{-\omega\sin(\omega t + \phi)\}\\ &= -A\omega^2\sin(\omega t + \phi).\end{aligned}$$

But $A\sin(\omega t + \phi)$ is just x, so

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -A\omega^2\sin(\omega t + \phi) = -\omega^2 x. \quad \blacksquare$$

Question 30 Find the first and second derivatives of x with respect to t for: (a) $x(t) = 4\sin(5t + \pi/4)$, (b) $x(t) = A\cos(\omega t)$. $\blacksquare$

14 Areas under curves, and integrals

Previous sections of this *Maths Handbook* have described how the gradient of a graph of y versus x has a special physical significance: it tells the rate of change of the y-variable as the x-variable changes. The gradient can be computed by graphical methods or by differentiation.

The *area 'under' a graph* of y versus x, meaning the area enclosed by (i) the plotted curve, (ii) the x-axis, and (iii) two boundaries formed by vertical lines at two particular values of x, also often has physical significance. For example, if we have a graph of the speed of an object as a function of time, the area under the curve indicates the distance travelled by the object between two specified values of time. This result probably isn't clear to you yet — it will be explained below — but it will give you an idea of why it is useful in physics to study the area under a graph.

Begin by considering a body moving with constant speed υ. The speed is given by the distance Δs travelled in some time interval Δt, divided by the time interval: $\upsilon = \Delta s/\Delta t$. If we know the speed, but not the distance travelled, we can nevertheless calculate the distance travelled from the rearranged equation: $\Delta s = \upsilon\,\Delta t$.

Now consider a body moving with non-uniform speed, whose speed versus time graph is shown in Figure 23a. If we consider a brief time interval Δt_1 from $t = t_A$ to $t = t_A + \Delta t$ during which the speed changes very little, then we can *approximate* the distance travelled as $\upsilon_1\,\Delta t_1$. The subscript '1' on the speed indicates that we want the speed during the time interval Δt_1. The distance Δs_1 travelled during this time interval is $\Delta s_1 \approx \upsilon_1\,\Delta t_1$.

Likewise, the distance Δs_2 travelled during the time interval Δt_2 from $t = t_A + \Delta t$ to $t = t_A + 2\Delta t$ is just $\Delta s_2 \approx \upsilon_2\,\Delta t_2$. To calculate the total distance travelled between $t = t_A$ and

$t = t_B$, we simply add the distance travelled during each brief time interval; Figure 23a divides this into six intervals, so

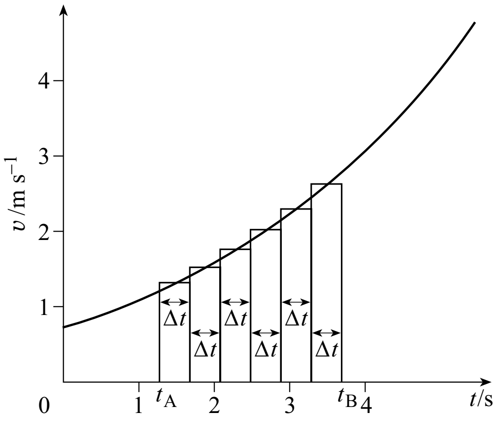
$$\Delta s = \Delta s_1 + \Delta s_2 + \Delta s_3 + \Delta s_4 + \Delta s_5 + \Delta s_6$$

and thus

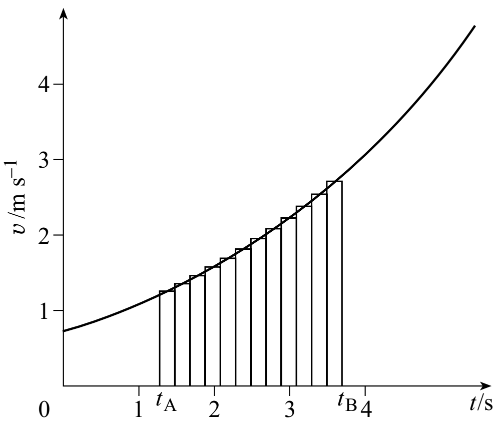
$$\Delta s \approx \upsilon_1\,\Delta t_1 + \upsilon_2\,\Delta t_2 + \upsilon_3\,\Delta t_3 + \upsilon_4\,\Delta t_4 + \upsilon_5\,\Delta t_5 + \upsilon_6\,\Delta t_6.$$

Using *summation notation*, we could write this as

$$\Delta s \approx \sum_{i=1}^{6} \upsilon_i\,\Delta t_i\,.$$



(a)



(b)

Figure 23 A graph of speed versus time, with the interval between $t = t_A$ and $t = t_B$ divided into (a) six time intervals of uniform width Δt and (b) into twelve finer time intervals.

Note, however, that this is just the area under the curve, i.e. the area enclosed by (i) the speed curve, (ii) the x-axis, and (iii) boundaries at $t = t_A$ and $t = t_B$. That is, the area under the speed versus time curve, calculated over the time interval from t_A to t_B, gives the distance travelled by the body during that interval.

The calculation of the area given above is not perfect because the area $\upsilon\,\Delta t$ of each rectangle of height υ and width Δt in Figure 23a is only an approximation to the area under the curve. It would be exact if the speed increased linearly with time, but because the graph is curved, the approximation is imperfect. However, the approximation can be made better by choosing smaller time intervals, as in Figure 23b, where 12 intervals are used instead of 6. In fact, as still smaller time intervals are chosen, the approximation becomes even better, and it becomes exact in the limit where Δt shrinks to zero. Just as for differentiation, where the notation is changed in the limit where intervals shrink to zero, we can replace the summation with a new symbol:

$$\Delta s \approx \sum_{i=1}^{6} \upsilon_i\,\Delta t_i \quad \text{becomes} \quad \Delta s = \int_{t_A}^{t_B} \upsilon(t)\;\mathrm{d}t \tag{14.1}$$

where

$$\int_{t_A}^{t_B} \upsilon(t)\,\mathrm{d}t = \lim_{\Delta t \to 0} \sum_i \upsilon_i \,\Delta t_i \qquad (14.2)$$

where $\upsilon(t)$ indicates that the speed is a function of time, i.e. not constant,

dt indicates that we are considering the limit of infinitesimal time intervals,

and $\int_{t_A}^{t_B}$ indicates that we sum over the time interval from $t = t_A$ to $t = t_B$.

This expression is referred to as 'the *definite integral* of υ with respect to t from $t = t_A$ to $t = t_B$'.

Note that the area under a curve is a *signed* quantity, in that areas that lie below the horizontal axis are regarded as negative. It follows that definite integrals may also be positive or negative.

More about integration

Although what follows in this box is not required for studying S207 *The Physical World*, it may help you to appreciate the use and usefulness of integrals if a few additional comments are made.

Integration, the technique of forming *integrals*, can be performed directly on equations without needing to consider graphs.

Integration can be regarded *almost* as the inverse of differentiation; if the derivative of f with respect to t is f', then the integral of f' with respect to t is $f + k$, where k is a constant. It is because of the need for the constant k that integration and differentiation are not *exact* inverses.

We can distinguish between a *definite integral*, which is an integration between two specified limits (e.g. $t = t_A$ and $t = t_B$ as above), and an *indefinite integral* in which the *equation* of the integral is found without reference to specified limits. As an example, recall from Section 11 that the derivative of x^2 with respect to x is $2x$. The *indefinite* integral of $2x$ with respect to x is therefore $x^2 + k$. The *definite* integral of $2x$ with respect to x over the interval from $x = 2$ to $x = 6$, say, can be found by *evaluating* the indefinite integral at both limits, and subtracting the first from the second. That is,

$$\int_{x=2}^{x=6} 2x\,\mathrm{d}x = [x^2 + k]_{x=6} - [x^2 + k]_{x=2} = 6^2 + k - 2^2 - k = 32$$

where the subscript on $[x^2 + k]_{x=6}$ indicates that the function in brackets should be evaluated at the stated x-value.

15 Differential equations

In Section 1, you saw how to rearrange equations involving some number of variables so that the variable of interest is isolated on the left-hand side. If an equation contains terms in both x and x^2, it is not possible to solve for x by this means, and it becomes necessary to use the solution for quadratic equations. Another form of equation that defies a simple algebraic solution is when a quantity and its derivative both appear together. Such an equation is called a *differential equation*.

In physics, several common differential equations are encountered. One is where the decay rate of a radioactive sample is proportional to the amount present: *QPM* page 123

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -ky.$$

A second example is simple harmonic motion, in which the force on an object is proportional to its displacement from an equilibrium position: *DM* page 112

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx.$$

Many differential equations, including most that you will encounter in this course, have *general solutions* that can be applied when needed. General solutions are functions (usually but not always expressed as an equation) that always satisfy the differential equation, but which necessarily contain extra parameters whose values can only be determined by reference to the specific problem at hand. If a differential equation contains only first derivatives, then the general solution will contain just one extra parameter compared to the original problem. The general solution for a differential equation involving second derivatives will require two additional parameters. For example, the general solution for the radioactive decay problem has already been met in Section 12 on the *exponential function*. You can verify that $y = a\mathrm{e}^{-kt}$ is a solution by substituting this into the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = -ky$, recalling from Equation 12.1 that the derivative of e^{kt} with respect to t is $k\mathrm{e}^{kt}$. Note that $y = a\mathrm{e}^{-kt}$ has one additional parameter, a, compared to the original differential equation, so this is the general solution of Equation 12.1.

Additional information is required to determine the values of the extra parameters in the general solution and thus to convert the general solution into a *particular solution* for a particular problem. For the radioactive decay problem, this would involve finding the value of a. The additional information often comes in the form of *initial conditions* or *boundary conditions* that describe the state of the physical system at some particular time or place. For the decay problem, knowledge of the decay rate (e.g. from Geiger counter measurements) at one time would provide enough information to determine the value of a in terms of k (which is specified as part of the problem).

The general solution for simple harmonic motion $m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx$ is $x = A\sin(\omega t + \phi)$. You can verify that this solves the differential equation, provided $\omega = \sqrt{k/m}$, by differentiating $x = A\sin(\omega t + \phi)$ twice as you did in Section 13. (Note the appearance of two additional parameters in the general solution, A and ϕ, as expected for a differential equation containing second derivatives.) Finding a particular solution for x in this case would require knowledge of k and m (specified as part of the problem) to give ω, plus two initial and/or boundary conditions capable of giving the amplitude A of the oscillation and the initial phase ϕ.

16 Vectors

A *vector* is a quantity that has both magnitude and direction, such as the velocity of a body. In contrast, a *scalar* has magnitude only. For example, temperature is a scalar variable; it may vary from point to point within a room, but there is no direction associated with each measurement. Velocity, on the other hand, is a vector variable; the velocity of a gas molecule measured at some point in a room has both speed and direction.

A vector may be represented diagramatically by an arrow, the length of which specifies the vector's magnitude and the direction of which is the same as the vector's direction. By convention, vectors are printed as bold symbols, e.g. $\boldsymbol{r}$, while the magnitude is written normally, e.g. r.

Handwritten vector symbols are written with a wavy underline, e.g. $\underset{\sim}{r}$ (which in the printing trade means 'make bold').

To specify a vector fully, both its magnitude (which is always positive) and its direction must be stated, e.g. '$\boldsymbol{F}$ is a force of 10 N acting vertically downwards'. The magnitude of $\boldsymbol{F}$ may be written as

$$F = |\boldsymbol{F}| = 10\,\mathrm{N}.$$

The modulus sign $|\boldsymbol{F}|$ provides an alternative way of indicating a magnitude.

16.1 Vector components

Any vector $\boldsymbol{a}$ in 3-dimensional space can be resolved into three mutually perpendicular (*Cartesian*) *components* a_x, a_y and a_z, given by

$$a_x = a\cos\theta_x$$

$a_y = a \cos \theta_y$

$a_z = a \cos \theta_z$,

where θ_x, θ_y and θ_z are respectively the angles between the direction of vector $\boldsymbol{a}$ and the x-, y- and z-axes and a is the magnitude of $\boldsymbol{a}$. The components are written as an ordered set in brackets, e.g.

$$\boldsymbol{a} = (a_x, a_y, a_z). \tag{16.1}$$

The *magnitude* of $\boldsymbol{a}$ is given by

$$|\boldsymbol{a}| = a = \sqrt{a_x^2 + a_y^2 + a_z^2}. \tag{16.2}$$

In many physics problems, only two dimensions are required, so only two components are considered, as in Figure 24, where the vector $\boldsymbol{f} = (f_x, f_y)$ is illustrated.

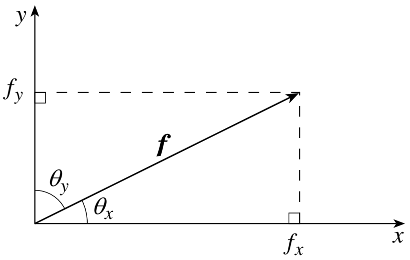


Figure 24 Any two dimensional vector $\boldsymbol{f}$ is characterized by two components, f_x and f_y, found by projecting perpendiculars to the x- and y-axes.

16.2 Addition and subtraction of vectors

For the *rare case* of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ having the *same* direction, addition is easy: the *resultant vector* $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$ is also in the same direction and of magnitude $c = a + b$. However, for the more common case of vectors with *different* directions, this simple rule does not apply, and addition must be carried out using graphical methods, shown in Figure 25 for two-dimensional vectors, or in terms of components.

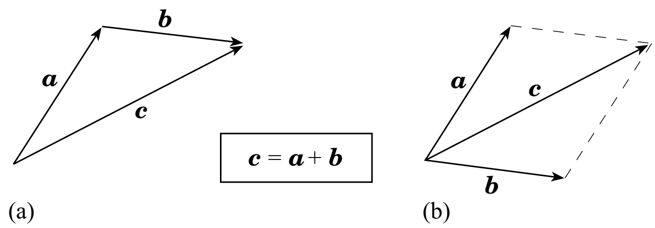


Figure 25 Equivalent methods of adding two two-dimensional vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, graphically: (a) triangle addition: (b) parallelogram addition. To sum more than two vectors, repeat application of the triangle rule.

Knowledge of the components greatly simplifies the addition, since

$$\text{if } \boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}, \text{ then } c_x = a_x + b_x,\ c_y = a_y + b_y, \text{ and } c_z = a_z + b_z. \tag{16.3}$$

Unless $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same direction, $|\boldsymbol{c}| \neq |\boldsymbol{a}| + |\boldsymbol{b}|$.

Similarly,

$$\text{if } \boldsymbol{c} = \boldsymbol{a} - \boldsymbol{b}, \text{ then } c_x = a_x - b_x,\ c_y = a_y - b_y, \text{ and } c_z = a_z - b_z. \tag{16.4}$$

Note that a vector can always be resolved into ‘component vectors’ along arbitrary directions at right angles to each other. In S207, vectors are normally specified in terms of scalar components tied to the x-, y-, z-coordinate axes.

16.3 Position and displacement vectors

Vectors are frequently used to specify the positions of points of interest. In three dimensions, the position of a point can be specified by giving its position coordinates (x, y, z), as shown in Figure 26a, but alternatively we can specify the vector $\boldsymbol{r}$ from the

origin to the point (x, y, z). Since the components of $\boldsymbol{r}$ are (x, y, z), the shorthand for this is $\boldsymbol{r} = (x, y, z)$. The vector $\boldsymbol{r}$ is known as the *position vector* of the point (x, y, z), and its magnitude is equal to the distance of the point from the origin:

$$r = \sqrt{x^2 + y^2 + z^2}.$$

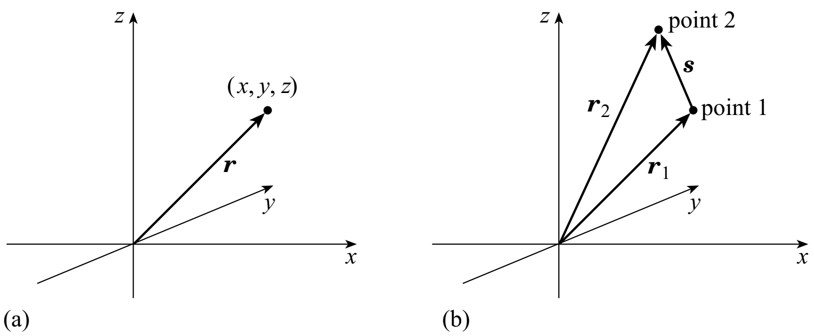


Figure 26 (a) The position vector $\boldsymbol{r}$ defines the position of a point relative to the origin. (b) The displacement vector $\boldsymbol{s}$ defines the *difference* in position for two points with position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, where $\boldsymbol{s} = \boldsymbol{r}_2 - \boldsymbol{r}_1$.

It is sometimes more convenient to specify the position of a point relative to another point not necessarily at the origin. In Figure 26b, the position of point 2 relative to point 1 is described by the vector $\boldsymbol{s}$, which is known as a *displacement vector*. Since point 1 has position vector $\boldsymbol{r}_1$ and point 2 has position vector $\boldsymbol{r}_2$ the triangle rule for addition of vectors tells us that

$$\boldsymbol{r}_1 + \boldsymbol{s} = \boldsymbol{r}_2$$

or

$$\boldsymbol{s} = \boldsymbol{r}_2 - \boldsymbol{r}_1.$$

In general, a displacement vector is the *difference* between two position vectors.

16.4 Multiplication and division of a vector by a scalar

It is straightforward to multiply a vector by a scalar. If $\boldsymbol{r} = a\boldsymbol{s}$, then the resulting vector $\boldsymbol{r}$ is in the same direction as the original $\boldsymbol{s}$ but with magnitude differing by a factor a. Similarly, the components of $\boldsymbol{r}$ are scaled from those of $\boldsymbol{s}$ by the factor a.

$$\text{If } \boldsymbol{r} = a\boldsymbol{s}, \text{ then } r_x = as_x,\ r_y = as_y,\ r_z = as_z. \tag{16.5}$$

If a is negative, then $\boldsymbol{r}$ will point in the opposite direction to $\boldsymbol{s}$.

For example, Newton's second law implies that a force (vector) applied to an object produces an acceleration (vector), with mass (scalar) as the constant of proportionality:

$$\boldsymbol{F} = m\boldsymbol{a}.$$

Since the mass m is a positive number, the force $\boldsymbol{F}$ is in the same *direction* as the acceleration $\boldsymbol{a}$, but its *magnitude* is different by a factor m.

We can also divide a vector by a scalar, since that is equivalent to multiplying by the reciprocal of a scalar, which is also a scalar. Rearranging the equation given above for Newton's second law,

$$\boldsymbol{a} = \frac{\boldsymbol{F}}{m} = \frac{1}{m}\boldsymbol{F}.$$

16.5 Unit vectors

It is often useful to divide a vector $\boldsymbol{r}$ by its own magnitude, to produce a *unit vector* $\hat{\boldsymbol{r}}$ defined as

$$\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{r}. \tag{16.6}$$

$\hat{\boldsymbol{r}}$ points in the same direction as $\boldsymbol{r}$, but has magnitude 1, i.e. $|\hat{\boldsymbol{r}}| = 1$. Note that $|\hat{\boldsymbol{r}}|$ is *unitless*.

16.6 Multiplication of two vectors

There are two completely different ways of multiplying two vectors: one produces a scalar, the other a vector.

The scalar product

The scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$ (also called the 'dot product') of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is a scalar equal to the product of their magnitudes multiplied by the cosine of the angle between their directions. Following Figure 27,

$$\boldsymbol{a} \cdot \boldsymbol{b} = ab\cos\theta. \tag{16.7}$$

An alternative expression, useful if the components of $\boldsymbol{a}$ and $\boldsymbol{b}$ are known, is

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_x b_x + a_y b_y + a_z b_z. \tag{16.8}$$

Combining these equations, we can also express the angle between the vectors as

$$\cos\theta = \frac{a_x b_x + a_y b_y + a_z b_z}{ab},$$

where the value of θ is found from the inverse cosine (arccosine) function.

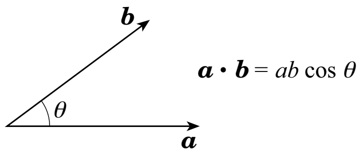


Figure 27 The scalar product.

The vector product

The *vector product* $\boldsymbol{a} \times \boldsymbol{b}$ (also called the 'cross product') of $\boldsymbol{a}$ and $\boldsymbol{b}$ is a vector with magnitude equal to the product of the magnitudes of $\boldsymbol{a}$ and $\boldsymbol{b}$ multiplied by the sine of the angle between them. The direction of the vector product is given by the *right-hand rule*, as illustrated in Figure 28b.

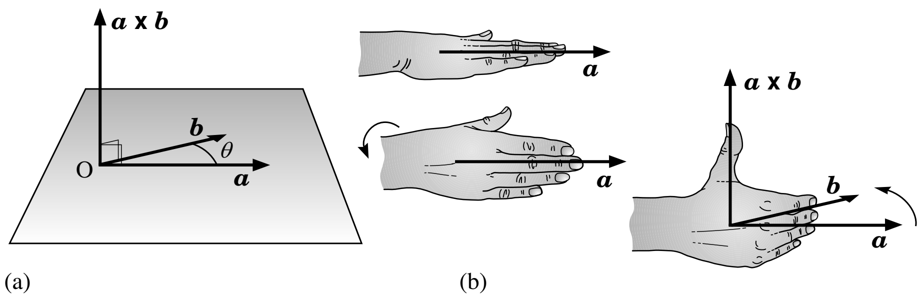


Figure 28 (a) The vector product. (b) The right-hand rule defines the direction of the vector product of two vectors. The palm and outstretched fingers and thumb of the right hand are aligned with the first vector $\boldsymbol{a}$, until the fingers can be bent to the direction of the second vector $\boldsymbol{b}$. The outstretched thumb then points in the direction of the vector product $\boldsymbol{a} \times \boldsymbol{b}$.

> $\boldsymbol{a} \times \boldsymbol{b}$ is a vector with magnitude $ab\sin\theta$ and direction perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ as given by the right-hand rule.

In terms of the components of the vectors,

$$\boldsymbol{a} \times \boldsymbol{b} = (a_y b_z - a_z b_y,\ a_z b_x - a_x b_z,\ a_x b_y - a_y b_x). \tag{16.9}$$

Note that the order of the vectors is unimportant in forming the scalar (dot) product, because $\cos(-\theta) = \cos\theta$ so $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$. However the order of the vectors is crucial in forming the vector (cross) product because $\sin(-\theta) = -\sin\theta$, so $\boldsymbol{b} \times \boldsymbol{a} = -\boldsymbol{a} \times \boldsymbol{b}$.

There is no mathematical operation defined as division *by* a vector, and so expressions such as $\boldsymbol{a}/\boldsymbol{b}$ or $a/\boldsymbol{b}$ are meaningless and should never be written.

Question 31 By considering the definition of the scalar and vector products, evaluate: (i) $\boldsymbol{a} \cdot \boldsymbol{a}$, (ii) $\boldsymbol{a} \times \boldsymbol{a}$. ■

16.7 Differentiation of vectors

Differentiation is not restricted to scalar functions; it can include vectors as well. If $\boldsymbol{r} = (x, y, z)$ then the derivative of $\boldsymbol{r}$ with respect to t is

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right).$$

Note furthermore that the derivative of a vector is also a vector, so has direction as well as magnitude. It is possible to imagine cases where r = constant, but $\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \neq 0$, i.e. where the magnitude of the vector is constant, but the direction changes.

17 Coordinate systems

The position of a point in space may be described by reference to a set of perpendicular x-, y-, z-axes ('*Cartesian axes*'), as shown in Figure 29. The (x, y, z) values of a point are called its *Cartesian coordinates*. These are also the components of the point's position vector $\boldsymbol{r} = (x, y, z)$. In two dimensions, the position vector is $\boldsymbol{r} = (x, y)$.

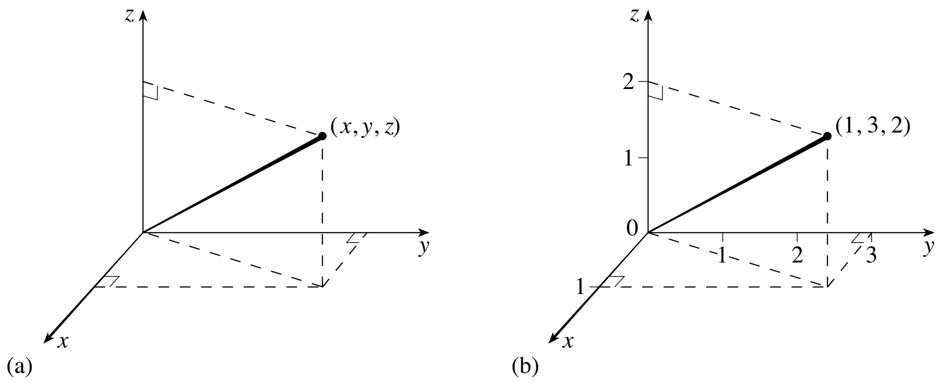


Figure 29 (a) The position of a point is specified by its x-, y- and z-coordinates. (b) The point for which $x = 1$, $y = 3$, $z = 2$ has coordinates (1, 3, 2).

In two dimensions, an alternative to Cartesian coordinates $\boldsymbol{r} = (x, y)$ is plane polar coordinates $\boldsymbol{r} = [r, \theta]$ (see Figure 30). Two numbers are still required to locate a point, but now the distance r and direction θ are specified rather than the x- and y- values. (We have also chosen to use square brackets around the pair of polar coordinates to help distinguish them from the pair of Cartesian coordinates, though this practice is neither essential nor universally adopted.) The units attached to the two coordinate pairs also differ; Cartesian coordinates have units of length, whereas plane polar coordinates have units of length and angle.

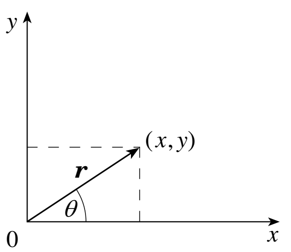


Figure 30 Cartesian coordinates (x, y) and plane polar coordinates $[r, \theta]$.

Coordinate transformations allow you to convert the coordinates expressed in one system to another system, albeit for the same point in space. To convert two-dimensional Cartesian coordinates to plane polar coordinates, the transformations are:

$$r = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x} \tag{17.1}$$

where θ is the angle between the x-axis ($\theta = 0$) and $\boldsymbol{r}$. Conversely, to convert plane polar coordinates to two-dimensional Cartesian coordinates:

$$x = r\cos\theta \qquad y = r\sin\theta. \tag{17.2}$$

Many other coordinate systems can be devised: Cartesian axes do not provide the only possible description of a location in space. For example, locations near the Earth's surface are often described in terms of latitude, longitude, and height above mean sea-level. In physics, Cartesian coordinates, whilst valid, often are not the most *convenient* ones to use, especially where rotational symmetry exists. Then, cylindrical polar coordinates are often much more useful. In the more restricted case of spherical symmetry, spherical polar coordinates are even more useful.

In spherical polar coordinates, the position of a point in space is described by reference to a distance coordinate r and two angular coordinates θ and ϕ, as shown in Figure 31. From the figure,

$$r^2 = x^2 + y^2 + z^2 \tag{17.3}$$

$$\cos\theta = \frac{z}{r} \tag{17.4}$$

$$\sin\phi = \frac{y}{r\sin\theta} \tag{17.5}$$

and

$$x = r\sin\theta\cos\phi \tag{17.6}$$

$$y = r\sin\theta\sin\phi \tag{17.7}$$

$$z = r\cos\theta. \tag{17.8}$$

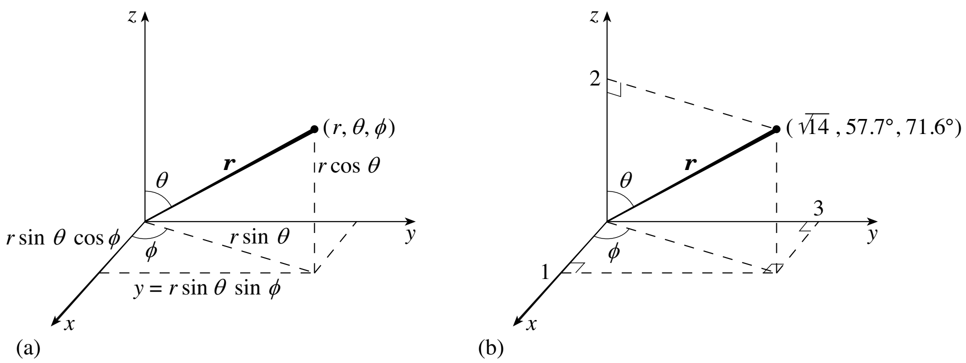


Figure 31 (a) In the spherical polar system, the position of a point is specified by its r, θ and ϕ coordinates. (b) The point for which $x = 1$, $y = 3$, $z = 2$ has spherical polar coordinates $(\sqrt{14}, 57.7°, 71.6°)$.

18 Circles and ellipses

The equation of a *circle* of radius r in two-dimensional Cartesian coordinates (x, y) is

$$x^2 + y^2 = r^2. \tag{18.1}$$

An *ellipse*, like a circle or a parabola, is a member of the family of curves known as *conic sections*. In a two-dimensional Cartesian coordinate system, an ellipse may be defined by choosing two lengths a and b, and then plotting all the points (x, y) that satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{18.2}$$

where a is greater than or equal to b. Such an ellipse is said to have a *semimajor axis* of length a, and a *semiminor axis* of length b.

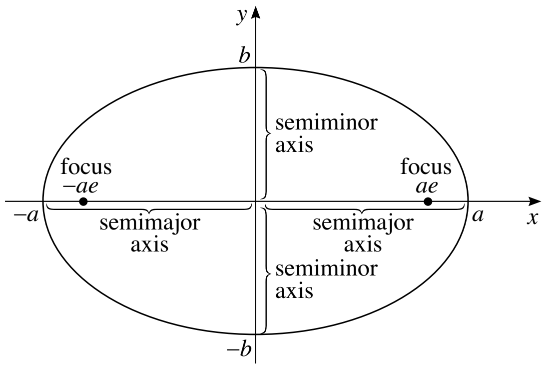


Figure 32 General features of an ellipse.

As you can see from Figure 32, an ellipse looks a bit like a partially squashed circle. A measure of how much the ellipse differs from a circle is given by its eccentricity e, which is defined as

$$e = \frac{1}{a}\sqrt{a^2 - b^2} \qquad \text{(for } a \text{ greater than or equal to } b\text{).} \tag{18.3}$$

The smaller the eccentricity, the more circular the ellipse. A circle is actually a special case of an ellipse with $a = b$, and therefore $e = 0$.

An ellipse has two special points on its semimajor axes, each called a *focus* (plural *foci*). These are located at the points (ae, 0) and ($-ae$, 0). One of the things that makes the foci special is that the sum of the two distances from any point on the ellipse to the two foci is a constant, equal to $2a$. An ellipse can be drawn, therefore, by placing a loop of inextensible string around two pins (the foci) and a pen (for drawing the ellipse), and moving the pen while keeping the string taut (see Figure 33). Try it! The semimajor axis a will be one-half the length of the string, and the eccentricity will be the separation of the pins divided by $2a$.

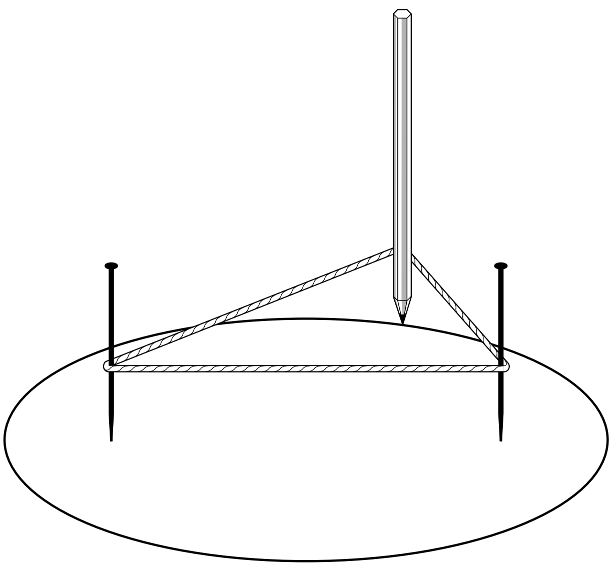


Figure 33 An ellipse can be drawn by placing a loop of string around two pins and a pen, and moving the pen while keeping the string taut.

Another quantity sometimes encountered is the *ellipticity*, given by

$$\text{ellipticity } \varepsilon = 1 - \frac{b}{a}. \tag{18.4}$$

Thus a circle, for which $b = a$, has an ellipticity of zero, whereas a highly elongated ellipse has ε approaching 1.

19 Probability

The physical world can be surprisingly unpredictable, and sometimes, rather than saying which of several outcomes will definitely happen, we are limited to talking about the *probability* of an outcome.

We concentrate here on simple cases, where all the possible outcomes are judged to be equally likely. For example, when a coin is tossed, the possible outcomes are heads or tails, and there is no reason to expect that one is more likely than the other. When a die ('die' is the singular of 'dice') is rolled we could score one, two, three, four, five, or six, and would expect all these outcomes to be equally likely (the die being taken as fair and unloaded).

The *probability* of a given outcome is defined to be the fraction of times that outcome is expected to happen in the long run. For example, the probability of tossing heads is 1/2 and the probability of tossing tails is also 1/2. In general, a probability of 0 corresponds to impossibility, and a probability of 1 corresponds to inevitability. The interesting area lies between these two extremes, where outcomes are uncertain. The closer the probability is to 1, the more likely it is to happen.

For a coin there are two alternative outcomes, each with a probability of 1/2. For a die there are six alternative outcomes, each with a probability of 1/6. In each case the sum of all the probabilities is 1. This can be expressed as a completely general rule:

> ***The normalization rule for probabilities***
>
> The sum of the probabilities of all the alternative outcomes is equal to 1.

The concept of probability is a theoretical one, based on our best estimate of the chances of several things happening. Strictly speaking, no experiment measures a probability: what is measured is the fraction of times an outcome occurs in a finite set of attempts: this is known as the *fractional frequency* of the outcome. In the long run, the measured fractional frequency is expected to approach the theoretical probability. Nevertheless, when dealing with uncertain outcomes there can be no cast-iron guarantees. You *could* toss a fair coin 100 times and get heads on every single toss.

There are many common misconceptions about probabilities. One is related to the (correct) expectation that everything should even out in the long run. Knowing this, many people believe that chance must actively conspire to help bring this about. If the first 100 tosses of a coin have produced 60 heads and 40 tails, they believe that they are owed a surplus of tails, so the next toss is more likely to result in a tail. This is not true. Although, in the extremely long run, the imbalance of heads and tails is expected to become negligible, *the coin has no memory* so, on each toss, heads and tails are equally likely, irrespective of any previous history.

So far, we have looked at probabilities of individual events, such as those associated with tossing a single coin or rolling a single die. Suppose we toss a coin *and* roll a die. What is the probability of tossing heads *and* rolling four? The answer is found by *multiplying* the probability of getting heads (1/2) by the probability of getting a four (1/6) to obtain

$$\tfrac{1}{2} \times \tfrac{1}{6} = \tfrac{1}{12}.$$

More generally, we have

> ***The multiplication rule for probabilities***
>
> If a number of outcomes occur independently of one another, the probability of them all happening together is found by *multiplying* their individual probabilities.

Example 19.1 If you throw three dice, what is the probability of scoring six on all of them?

The probability of getting a six on one die is $\frac{1}{6}$, so the probability of getting a six on all the dice is

$$\tfrac{1}{6} \times \tfrac{1}{6} \times \tfrac{1}{6} = (\tfrac{1}{6})^3 \approx 4.63 \times 10^{-3} \text{ to 3 significant figures.} \quad \blacksquare$$

Note that although we must curtail the number of significant figures in the answer, the number 1/6, in the probability of throwing a six on one die, is exact if the die is fair.

Probabilities can also be added together. This is appropriate when there are a number of alternative (mutually exclusive) outcomes, such as rolling either a four or five on a single throw of a die.

The addition rule for probabilities

If a number of alternative outcomes are mutually exclusive, the probability of getting *one or other* of these outcomes is found by *adding* their individual probabilities.

Example 19.2 What is the probability of rolling a four or a five on a single throw of a die?

The probability of rolling a four is $\frac{1}{6}$, the probability of rolling a five is also $\frac{1}{6}$, so the probability of rolling a four *or* a five is

$$\tfrac{1}{6}+\tfrac{1}{6}=\tfrac{1}{3}. \quad \blacksquare$$

The normalization rule, quoted earlier, can therefore be interpreted as expressing the obvious fact that we are *certain* to get one or other of the possible outcomes.

Question 32 (a) If you toss an ordinary 10p coin ten times, what is the probability that you will get 'heads' on every toss?

(b) If you draw one playing card from a standard pack of 52, what is the probability of that card being either the Ace, King or Queen of Spades? ■

20 Answers to questions

Note: These solutions have been written out using many steps as an aid to your understanding, but in many cases you may have been able to arrive at the answer in fewer steps. In laying out calculations, you may include as many (or as few) steps as you feel comfortable with. Whether or not you choose to show as much detail as we have done here, we would encourage you to lay out your calculations carefully and to explain your working.

When the final answer can be written in several alternative forms, we have endeavoured to give all the alternatives in the answers that follow, but on some occasions there are several different ways of doing the working too, and we have only been able to give one method.

Q1 (i) $\frac{1}{2}(\upsilon_x + u_x)t = \frac{1}{2}\upsilon_x t + \frac{1}{2}u_x t$

$= \dfrac{\upsilon_x t}{2} + \dfrac{u_x t}{2}$.

(ii) $(a - 2b)^2 = (a - 2b)(a - 2b)$

$= \{a(a - 2b)\} - \{2b(a - 2b)\}$

$= a^2 - 2ab - 2ba + 4b^2$

$= a^2 - 4ab + 4b^2$.

Q2 (i) $\dfrac{(a-b)-(a-c)}{2} = \dfrac{(a-b-a+c)}{2} = \dfrac{c-b}{2}$.

(ii) $\dfrac{a^2 - b^2}{a+b} = \dfrac{(a+b)(a-b)}{a+b} = a - b$.

Q3 (i) $4a$.

(ii) $\sqrt{4a^2} = 2a$.

(iii) $4\sqrt{a^2} = 4a$.

(iv) $2(a + a) = 2 \times 2a = 4a$.

(v) $\dfrac{2a+6a}{2} = \dfrac{8a}{2} = 4a$.

(vi) $2a + \dfrac{6a}{2} = 2a + 3a = 5a$.

So (i), (iii), (iv) and (v) are equivalent.

Q4 (a) $h \times \dfrac{\upsilon}{\lambda} = \dfrac{h\upsilon}{\lambda}$ (multiplying the denominator by h).

(b) $\dfrac{h\upsilon}{h\lambda} = \dfrac{\upsilon}{\lambda}$ (cancelling by h).

(c) $\dfrac{\mu_0}{2\pi} \times \dfrac{i_1 i_2}{d} = \dfrac{\mu_0 i_1 i_2}{2\pi d}$ (multiplying two fractions together).

(d) $\dfrac{3x}{2t}/2 = \dfrac{3x}{4t}$ (dividing $\dfrac{3x}{2t}$ by 2).

(e) $\dfrac{2xy}{z} \div \dfrac{z}{2} = \dfrac{2xy}{z} \times \dfrac{2}{z} = \dfrac{4xy}{z^2}$ (dividing by a fraction).

(f) $\dfrac{2}{3} + \dfrac{5}{6} = \dfrac{2\times 6}{3\times 6} + \dfrac{5\times 3}{6\times 3} = \dfrac{12}{18} + \dfrac{15}{18} = \dfrac{27}{18} = \dfrac{27/9}{18/9} = \dfrac{3}{2}$ (adding two fractions).

(g) $\dfrac{a}{b} - \dfrac{c}{d} = \dfrac{ad}{bd} - \dfrac{cb}{db} = \dfrac{ad - cb}{bd}$ (subtracting one fraction from another).

Q5 (a) $E = hf$.

Dividing both sides by h gives $f = \dfrac{E}{h}$.

(b) $E = hf - \phi$.

Adding ϕ to both sides gives $hf = E + \phi$.

Dividing both sides by h gives $f = \dfrac{E+\phi}{h}$.

Q6 (a) $E = -\dfrac{GmM}{r}$.

Multiplying both sides by r gives

$$Er = -GmM.$$

Dividing both sides by $-G$ and M gives

$$m = -\frac{Er}{GM}.$$

(b) $E^2 = p^2c^2 + m^2c^4$.

Subtracting p^2c^2 from both sides gives

$$E^2 - p^2c^2 = m^2c^4.$$

Dividing both sides by c^4 gives

$$m^2 = \frac{E^2 - p^2c^2}{c^4}.$$

Taking the square root of both sides gives

$$m = \pm\sqrt{\frac{E^2 - p^2c^2}{c^4}}.$$

Since m is declared to be a mass, it must be positive so the negative square root is rejected on physical grounds.

Therefore $m = \sqrt{\dfrac{E^2 - p^2c^2}{c^4}}$.

Note that this could also have been written as

$$m = \frac{\sqrt{E^2 - p^2c^2}}{c^2}.$$

(c) $f = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}}$.

Squaring both sides gives $f^2 = \left(\dfrac{1}{2\pi}\right)^2 \dfrac{k}{m}$.

Multiplying both sides by m gives $mf^2 = \left(\dfrac{1}{2\pi}\right)^2 k$.

Dividing both sides by f^2 gives $m = \left(\dfrac{1}{2\pi}\right)^2 \dfrac{k}{f^2}$.

This could also have been written as

$$m = \left(\frac{1}{2\pi f}\right)^2 k, \quad m = \frac{k}{(2\pi f)^2} \quad \text{or}$$

$$m = \frac{k}{4\pi^2 f^2}.$$

Q7 (a) First rearrange $p = mv$ to give an expression for v:

$$v = \frac{p}{m}.$$

Then substitute this expression into $E_{\text{trans}} = \frac{1}{2}mv^2$ to give

$$E_{\text{trans}} = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\left(\frac{p}{m}\right)^2 = \frac{1}{2}\frac{mp^2}{m^2} = \frac{p^2}{2m}.$$

(b) First rearrange $n\lambda = d\sin\theta_n$ to give an expression for λ:

$$\lambda = \frac{d\sin\theta_n}{n}.$$

Then substitute this expression into $c = f\lambda$ to give

$$c = f\lambda = f \times \frac{d\sin\theta_n}{n} = \frac{fd\sin\theta_n}{n}.$$

Then rearrange this (by multiplying both sides by n and dividing both sides by $d\sin\theta_n$) to give

$$nc = fd\sin\theta_n$$

i.e. $f = \dfrac{cn}{d\sin\theta_n}$.

Q8 (i) $a - b = 1$; (I)

$a + b = 5$ (II)

From (I), $a = 1 + b$. (III)

Substituting into (II) $(1 + b) + b = 5$,

i.e. $2b = 4$, so $b = 2$.

Substituting into (III) gives

$$a = 1 + b = 1 + 2 = 3.$$

So the solution is $a = 3$ and $b = 2$.

(ii) $a^2 - b^2 = 8$; (I)

$a + b = 2$. (II)

From (II), $b = 2 - a$, (III)

Substituting into (I)

$$a^2 - (2 - a)^2 = 8$$

$$a^2 - (4 - 4a + a^2) = 8$$

$$a^2 - 4 + 4a - a^2 = 8$$

$$4a - 4 = 8$$

i.e. $4a = 12$, so $a = 3$.

Substituting into (III) gives

$$b = 2 - a = 2 - 3 = -1.$$

So the solution is $a = 3$, $b = -1$.

Q9 (a) The units on the left-hand side of $E = Gm_1m_2r$ are

$$\text{J} = \text{N m} = (\text{kg m/s}^2) \times \text{m} = \text{kg m}^2/\text{s}^2.$$

The units on the right-hand side of $E = Gm_1m_2r$ are

$$(\text{N m}^2/\text{kg}^2) \times \text{kg} \times \text{kg} \times \text{m} = \text{N m}^3$$

$$= (\text{kg m/s}^2) \times \text{m}^3 = \text{kg m}^4/\text{s}^2.$$

So the units on the left-hand side are not the same as the units on the right-hand side, so the equation must be wrong.

(b) The units on the left-hand side of $F = \dfrac{Gm_1m_2}{r^2}$ are

$$\text{N} = \text{kg m/s}^2.$$

The units on the right-hand side of $F = \dfrac{Gm_1m_2}{r^2}$ are

$$\begin{aligned}\frac{(\text{N m}^2/\text{kg}^2)\times\text{kg}\times\text{kg}}{\text{m}^2} &= \frac{\text{N m}^2}{\text{m}^2}\\ &= \text{N}\\ &= \text{kg m/s}^2.\end{aligned}$$

The units on the left-hand side of the equation are the same as the units on the right-hand side, so the equation is dimensionally correct.

Q10 (a) $10^2 \times 10^3 = 10^{2+3} = 10^5 = 100\,000$.

(b) $10^2/10^3 = 10^{2-3} = 10^{-1} = \frac{1}{10} = 0.1$

(c) $t^2/t^{-2} = t^{2-(-2)} = t^4$.

(d) $\sqrt{10^4} = (10^4)^{1/2} = 10^{4/2} = 10^2 = 100$.

(e) $(100)^{3/2} = (100^{1/2})^3 = (\sqrt{100})^3 = 10^3 = 1000$.

(f) $(125)^{-1/3} = \dfrac{1}{\sqrt[3]{125}} = \dfrac{1}{5} = 0.2$.

(g) $\sqrt{\dfrac{x^4}{4}} = \left(\dfrac{x^4}{4}\right)^{1/2} = \dfrac{x^{4/2}}{\sqrt{4}} = \dfrac{x^2}{2}$.

(h) $(2\,\text{kg})^2/(2\,\text{kg})^{-2} = (2^2/2^{-2})\,(\text{kg}^2/\text{kg}^{-2})$

$= (2^2 \times 2^2)\,(\text{kg}^2 \times \text{kg}^2) = 16\,\text{kg}^4$.

Q11 (a) $1467\,851 = 1.467\,851 \times 1000\,000$

$= 1.467\,851 \times 10^6$.

(b) $0.0046 = \dfrac{4.6}{1000} = \dfrac{4.6}{10^3} = 4.6\times10^{-3}$.

(c) $11 \times 10^6 = (1.1 \times 10^1) \times 10^6$

$= 1.1 \times 10^{1+6} = 1.1 \times 10^7$.

(d) $0.0031 \times 10^{-2} = (3.1 \times 10^{-3}) \times 10^{-2} = 3.1 \times 10^{-3+(-2)}$
$= 3.1 \times 10^{-5}$.

Q12 (a) $(3 \times 10^6) \times (7 \times 10^{-2})$

$= (3 \times 7) \times 10^{6+(-2)}$

$= 21 \times 10^4$

$= 2.1 \times 10^5$.

(b) $\dfrac{8\times10^4}{4\times10^{-1}} = \dfrac{8}{4}\times10^{4-(-1)} = 2\times10^5$.

(c) Collecting powers of ten in the numerator and denominator gives

$$\frac{10^4 \times (4 \times 10^4)}{0.001 \times 10^{-2}} = \frac{4 \times 10^{4+4}}{10^{-3} \times 10^{-2}}$$
$$= 4 \times 10^{8-(-3-2)}$$
$$= 4 \times 10^{13}.$$

Q13 (a) $\log_{10}1000 = 3$, since $1000 = 10^3$.

(b) $\log_{10}0.001 = -3$, since $0.001 = 10^{-3}$.

(c) $\log_{10}\sqrt{10} = \frac{1}{2}$, since $\sqrt{10} = 10^{1/2}$.

Q14 Using Equation 5.3, $3x^4 = 3 \times x^4$, so $\log_{10}(3x^4) = \log_{10}3 + \log_{10}x^4$.

Using Equation 5.5, $\log_{10}x^4 = 4\log_{10}x$.

Thus $\log_{10}(3x^4) = \log_{10}3 + 4\log_{10}x$.

Q15 (a) The speed is

$$\frac{1.2\,\text{km}}{65.6\,\text{s}} = 0.018\,\text{km s}^{-1}$$
$$= 1.8 \times 10^{-2}\ \text{km s}^{-1}$$

to two significant figures. (Distance is given to two significant figures and time is given to three significant figures, and the answer is given to the same number of significant figures as the least precisely known quantity.)

(b) The speed is

$$\frac{0.09\,\text{km}}{5.1\,\text{s}} = 0.02\,\text{km s}^{-1}$$
$$= 2 \times 10^{-2}\ \text{km s}^{-1}$$

to one significant figure.

(As in (a), the answer is given to the same number of significant figures as the least precisely known quantity. In this case, remembering that initial zeroes do not count as significant figures, the least precisely known quantity is distance.)

(c) Speed $= \dfrac{3000\,\text{km}}{0.01\,\text{s}} = 3 \times 10^5\ \text{km s}^{-1}$ to one significant figure.

(Convention could be seen to be suggesting that distance is known to four significant figures, i.e. it lies between 2999.5 km and 3000.5 km. The time, however, is quoted to just one significant figure, so the speed should be given to the same precision and therefore 3×10^5 km s^{-1} is the only unambiguous way to write the result.)

Q16 (a) $90° = \dfrac{360°}{4} = \dfrac{2\pi}{4}$ radians $= \dfrac{\pi}{2}$ radians;

$30° = \dfrac{\pi}{6}$ radians;

$180° = \pi$ radians.

(b) $\dfrac{\pi}{8}$ radians $= \dfrac{180°}{8} = 22.5°$;

$\dfrac{3\pi}{2}$ radians $= \dfrac{3 \times 180°}{2} = 270°$.

Q17 (a) (i) $\sin 49° \approx 0.75$ (using a calculator in degree mode).

(ii) $\cos\dfrac{\pi}{8} \approx 0.924$ (using a calculator in radian mode)

(iii) $\tan\dfrac{\pi}{4} = \tan 45° = 1$ (from Table 1).

(b) $\sin^{-1}(0.1) = \arcsin(0.1) \approx 0.100$ radians $= 5.74°$.

Q18 $\cos\alpha = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{12\,\text{m}}{13\,\text{m}} \approx 0.9231.$

So, $\alpha \approx 22.6° = 0.395$ radians.

Q19 See Figure 34.

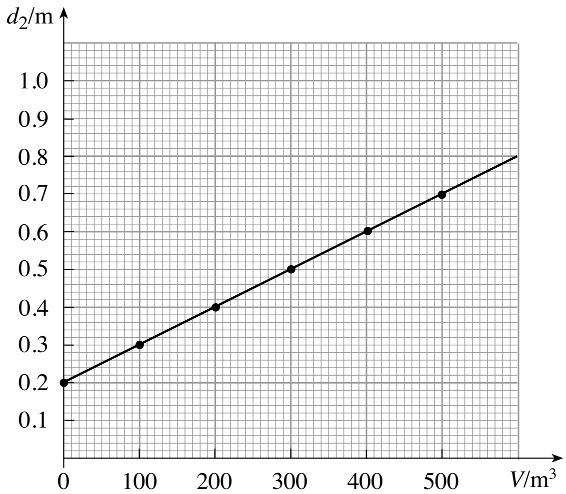


Figure 34 Answer to Question 19: A graph showing how the depth of water, d_2, in Pool 2 varies with the volume of water, V, that has been pumped into the pool.

Q20 Using points at $V = 100\,\text{m}^3$ and $V = 500\,\text{m}^3$ gives

rise $= \Delta d_2 = 0.7\,\text{m} - 0.3\,\text{m} = 0.4\,\text{m}$,

run $= \Delta V = 500\,\text{m}^3 - 100\,\text{m}^3 = 400\,\text{m}^3$.

Therefore

$$\text{gradient} = \frac{\text{rise}}{\text{run}}$$
$$= \frac{\Delta d_2}{\Delta V} = \frac{0.4\,\text{m}}{400\,\text{m}^3}$$
$$= 0.001\,\text{m}^{-2}.$$

So the depth of water in Pool 2 increases at one-half the rate at which the depth of water in Pool 1 increases. This is probably because Pool 1 has a larger surface area.

Q21 (a) is a straight line passing through the origin. Its gradient is 3.

(b) is not a straight line passing through the origin (though it is a straight line).

(c) is a straight line passing through the origin. Its gradient is m.

(d) is not a straight-line graph.

(e) is a straight line passing through the origin. Its gradient is $\frac{1}{2}m$.

Q22 (a) The gradient of the graph is $\frac{50-20}{5-0}=\frac{30}{5}=6$ and the intercept on the vertical axis is 20, so the equation is $a = 6b + 20$.

(b) The gradient of the graph is $\frac{25-(-5)}{10-0}=\frac{30}{10}=3$ and the intercept on the vertical axis is −5, so the equation is $r = 3s - 5$.

(c) The gradient of the graph is $\frac{0-10}{4-0}=-2.5$ and the intercept on the vertical axis is 10, so the equation is $q = -2.5p + 10$.

Q23 See Figure 35.

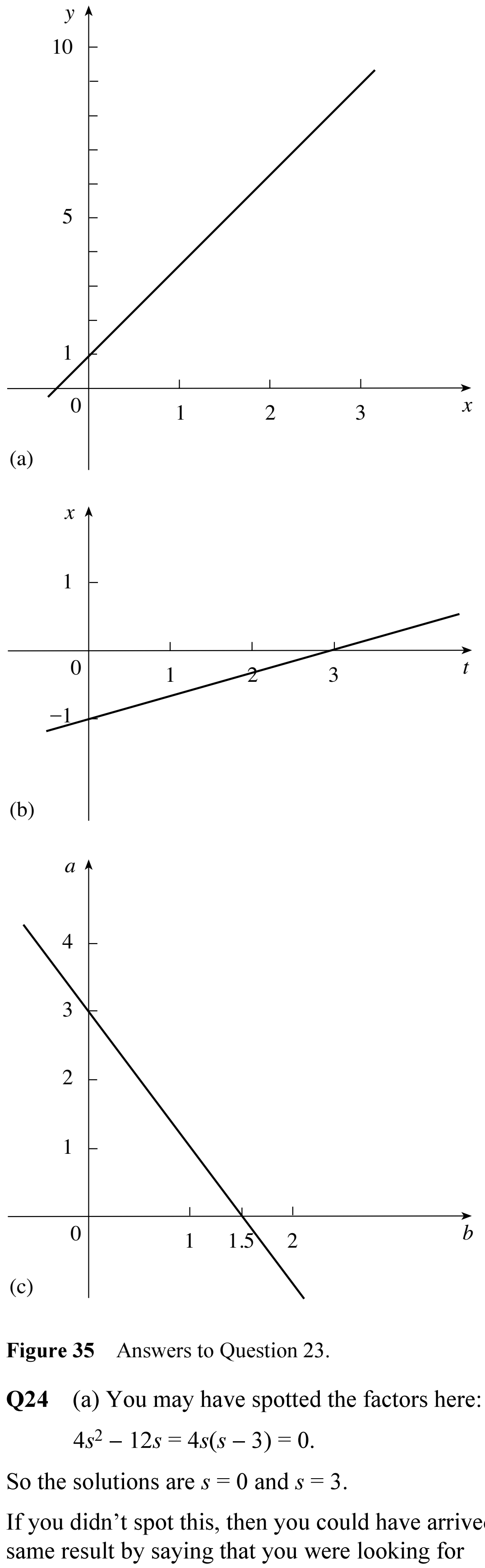


Figure 35 Answers to Question 23.

Q24 (a) You may have spotted the factors here:

$$4s^2 - 12s = 4s(s - 3) = 0.$$

So the solutions are $s = 0$ and $s = 3$.

If you didn't spot this, then you could have arrived at the same result by saying that you were looking for solutions with a sum of $-\frac{b}{a}=-\frac{(-12)}{4}=3$ and a product of $\frac{c}{a}=\frac{0}{4}=0$.

(b) Comparing $s^2 - s - 6 = 0$ with Equation 10.1 shows that $a = 1$, $b = -1$ and $c = -6$ on this occasion, so we are

looking for solutions whose sum is $-\frac{b}{a} = -\frac{(-1)}{1} = 1$ and whose product is $\frac{c}{a} = \frac{(-6)}{1} = -6$. Two possible solutions which meet these criteria are $s = 3$ and $s = -2$, and we can check these by multiplying out:

$$\begin{aligned}(s-3)(s+2) &= s(s+2) - 3(s+2) \\ &= s^2 + 2s - 3s - 6 \\ &= s^2 - s - 6.\end{aligned}$$

So $s^2 - s - 6 = 0$ when $(s - 3)(s + 2) = 0$, i.e. the solutions are $s = 3$ and $s = -2$.

Q25 (a) Comparing $s^2 - s - 6 = 0$ with Equation 10.1 shows that $a = 1$, $b = -1$ and $c = -6$ on this occasion, so the solutions are

$$\begin{aligned}s &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-1) \pm \sqrt{(-1)^2 - \{4 \times 1 \times (-6)\}}}{2 \times 1} \\ &= \frac{1 \pm \sqrt{1+24}}{2}.\end{aligned}$$

So $s = \frac{1+\sqrt{25}}{2} = \frac{1+5}{2} = \frac{6}{2} = 3$

or $s = \frac{1-\sqrt{25}}{2} = \frac{1-5}{2} = \frac{-4}{2} = -2$.

Checking the solutions:

For $s = 3$, $s^2 - s - 6 = (3)^2 - 3 - 6 = 9 - 3 - 6 = 0$.

For $s = -2$, $s^2 - s - 6 = (-2)^2 - (-2) - 6 = 4 + 2 - 6 = 0$.

So both solutions are correct. Note also that these are the same solutions obtained when you solved this equation by factorization in Question 24.

(b) Comparing $2y^2 + 5y + 1 = 0$ with Equation 10.1 shows that $a = 2$, $b = 5$ and $c = 1$ on this occasion, so the solutions are

$$\begin{aligned}y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-5 \pm \sqrt{5^2 - (4 \times 2 \times 1)}}{2 \times 2} \\ &= \frac{-5 \pm \sqrt{25-8}}{4}.\end{aligned}$$

So $\quad y = \frac{-5+\sqrt{17}}{4} = -0.2192$ to 4 sig figs

or $\quad y = \frac{-5-\sqrt{17}}{4} \approx -2.281$ to 4 sig figs.

Checking the solutions:

For $y = -0.2192$, $2y^2 + 5y + 1$
$= 2(-0.2192)^2 + 5(-0.2192) + 1 \approx 0$.

For $y = -2.281$, $2y^2 + 5y + 1$
$= 2(-2.281)^2 + 5(-2.281) + 1 \approx 0$.

So both solutions are correct to within the accuracy with which we have carried out the calculations.

Q26 (a) $x(t) = t^7$, so the n in Equation 11.3 is 7, that is

$$\frac{dx}{dt} = 7t^6.$$

(b) $y(x) = \frac{1}{x^3} = x^{-3}$, so the n in Equation 11.3 is -3, that is

$$\frac{dy}{dx} = -3x^{-4} = -\frac{3}{x^4}.$$

(c) $w(t) = 3\sqrt{t} = 3t^{1/2}$, so

$$\begin{aligned}\frac{dw}{dt} &= 3\frac{d}{dt}(t^{1/2}) \\ &= 3 \times \frac{1}{2}t^{-1/2} \\ &= \frac{3}{2} \times \frac{1}{t^{1/2}} \\ &= \frac{3}{2\sqrt{t}}.\end{aligned}$$

(d) $z(y) = 4y^2 + y$, so

$$\begin{aligned}\frac{dz}{dy} &= \frac{d}{dy}(4y^2) + \frac{d}{dy}(y) \\ &= (4 \times 2y^1) + (1 \times y^0) \\ &= 8y + 1.\end{aligned}$$

Q27 $x(t) = At^3 + Bt^2 + Ct + D$, so that

$$\begin{aligned}\frac{dx(t)}{dt} &= \frac{d}{dt}(At^3) + \frac{d}{dt}(Bt^2) \\ &\quad + \frac{d}{dt}(Ct) + \frac{d}{dt}(D) \\ &= (A \times 3t^2) + (B \times 2t^1) + (C \times 1t^0) + 0 \\ &= 3At^2 + 2Bt + C.\end{aligned}$$

$$\begin{aligned}\frac{d^2x(t)}{dt^2} &= \frac{d}{dt}\left(\frac{dx}{dt}\right) \\ &= \frac{d}{dt}(3At^2) + \frac{d}{dt}(2Bt) + \frac{d}{dt}(C) \\ &= (3 \times A \times 2t^1) + (2B \times 1t^0) + 0 \\ &= 6At + 2B\end{aligned}$$

Q28 $N(t) = N_0 e^{-t/\tau}$, so

$$\frac{dN}{dt} = \frac{d}{dt}(N_0 e^{-t/\tau}) = N_0 \frac{d}{dt}(e^{-t/\tau}).$$

From the definition of the exponential function,

$$\frac{d}{dt}(e^{-t/\tau}) = -\frac{1}{\tau}(e^{-t/\tau}), \text{ so}$$

$$\begin{aligned}\frac{dN}{dt} &= N_0 \frac{d}{dt}(e^{-t/\tau}) \\ &= -\frac{N_0}{\tau} e^{-t/\tau} \\ &= \frac{-N}{\tau}.\end{aligned}$$

For $\tau = 8100$ years, $N_0 = 1.0 \times 10^6$ and $t = 20\,000$ years, giving

$N = 1.0 \times 10^6 e^{-20\,000/8100} = 8.5 \times 10^4$ undecayed nuclei.

This result is consistent with Figure 18.

Q29 Reading from Figure 18, the time taken for the radioactivity to halve is approximately 5600 years.

We are told that τ is 8100 years, so $\tau_{1/2} = \tau \log_e 2 \approx 5600$ years which is in agreement with the value obtained from the graph.

Q30 (a) $x(t) = 4\sin(5t + \pi/4)$, so

$$\begin{aligned}\frac{dx}{dt} &= 4\frac{d}{dt}\{\sin(5t+\pi/4)\}\\ &= 4\times 5\cos(5t+\pi/4)\\ &= 20\cos(5t+\pi/4).\end{aligned}$$

$$\begin{aligned}\frac{d^2x}{dt^2} &= 20\frac{d}{dt}\{\cos(5t+\pi/4)\}\\ &= 20\times\{-5\sin(5t+\pi/4)\}\\ &= -100\sin(5t+\pi/4).\end{aligned}$$

(b) $x(t) = A\cos(\omega t)$, so

$$\begin{aligned}\frac{dx}{dt} &= A\times\frac{d}{dt}\{\cos(\omega t)\}\\ &= A\times\{-\omega\sin(\omega t)\}\\ &= -A\omega\sin(\omega t).\end{aligned}$$

$$\begin{aligned}\frac{d^2x}{dt^2} &= -A\omega\times\frac{d}{dt}\{\sin(\omega t)\}\\ &= -A\omega\times\{\omega\cos(\omega t)\}\\ &= -A\omega^2\cos(\omega t).\end{aligned}$$

But $A\cos(\omega t)$ is just x, so

$$\begin{aligned}\frac{d^2x}{dt^2} &= -A\omega^2\cos(\omega t)\\ &= -\omega^2 x.\end{aligned}$$

Q31 (i) $\boldsymbol{a}\cdot\boldsymbol{a} = a^2$ a positive scalar since $\cos 0 = 1$.

(ii) $\boldsymbol{a}\times\boldsymbol{a} = \mathbf{0}$, i.e. it vanishes since $\sin 0 = 0$. The vector (cross) product of any parallel vectors is always zero.

Q32 (a) The probability of getting heads on one toss is $\frac{1}{2}$, so the probability of getting heads every time is

$$\begin{aligned}\tfrac{1}{2}\times\tfrac{1}{2}\times\tfrac{1}{2}\times\tfrac{1}{2}\times\tfrac{1}{2}\times\tfrac{1}{2}\times\tfrac{1}{2}\times\tfrac{1}{2}\times\tfrac{1}{2}\times\tfrac{1}{2} &= \tfrac{1}{2^{10}}\\ &= 9.766\times 10^{-4}\end{aligned}$$

to 4 significant figures.

(b) The probability of drawing *just* the Ace of Spades is $\frac{1}{52}$, similarly for the other two cards, so the probability of the card being *either* the Ace, King or Queen of Spades is

$$\tfrac{1}{52}+\tfrac{1}{52}+\tfrac{1}{52} = \tfrac{3}{52}.$$

Mathematical signs and symbols

$=$	equals	$\pm$	plus or minus the following number
$\approx$	approximately equals	$\mp$	minus or plus, taken in the same order as a preceding $\pm$
$\sim$	is of order of (i.e. is less than 10 times bigger or smaller than)	Δx	the change in x
$\neq$	is not equal to	$f(x)$	a function f depending on the variable x
$>$	is greater than	$\lvert x \rvert$	the modulus or absolute value of a quantity (i.e. ignoring any – sign)
$\gg$	is much greater than	$\lvert \boldsymbol{a} \rvert$	the magnitude or length of a vector
$\geq$	is greater than or equal to (i.e. is no less than)	$\boldsymbol{a} \cdot \boldsymbol{b}$	the scalar (dot) product of two vectors
$<$	is less than	$\boldsymbol{a} \times \boldsymbol{b}$	the vector (cross) product of two vectors
$\ll$	is much less than	$\sum_{i=1}^{N} m_i$	the sum of $m_1 + m_2 + m_3 + \ldots + m_N$
$\leq$	is less than or equal to (i.e. is no more than)	$\langle x \rangle$	the average value of x
$\propto$	is proportional to	$\mathrm{d}y/\mathrm{d}t$, y', $\dot{y}$	the derivative of y with respect to t; = the gradient of y against t
∞	infinity	$\mathrm{d}^2y/\mathrm{d}t^2$, y'', $\ddot{y}$	the second derivative of y with respect to t
$\sqrt{x}$	the positive square root of x	$\int_{t_A}^{t_B} x(t)\,\mathrm{d}t$	the definite integral of the t-dependent function $x(t)$ with respect to t, evaluated over the interval from $t = t_A$ to $t = t_B$.
$\sqrt[n]{x}$	the nth root of x; $= x^{1/n}$		

Derivatives

[A, n, k and ω are constants;
x, y and z are functions of t]

x	$\dfrac{\mathrm{d}x}{\mathrm{d}t}$
A	0
t^n	nt^{n-1}
$\sin(\omega t)$	$\omega\cos(\omega t)$
$\cos(\omega t)$	$-\omega\sin(\omega t)$
e^{kt}	$k\mathrm{e}^{kt}$
Ay	$A\dfrac{\mathrm{d}y}{\mathrm{d}t}$
$y+z$	$\dfrac{\mathrm{d}y}{\mathrm{d}t}+\dfrac{\mathrm{d}z}{\mathrm{d}t}$

SI derived units and conversions

[The base units are: m, kg, s; A, K, cd, mol.]

Quantity	Unit	Conversion		
speed	m s^{-1}			
acceleration	m s^{-2}			
angular frequency	s^{-1}			
angular speed	rad s^{-1}			
angular acceleration	rad s^{-2}			
linear momentum	kg m s^{-1}			
angular momentum	kg m^2 s^{-1}			
force	newton (N)	1 N	=	1 kg m s^{-2}
energy	joule (J)	1 J	=	1 N m = 1 kg m^2 s^{-2}
torque	N m			
power	watt (W)	1 W	=	1 J s^{-1}
pressure	pascal (Pa)	1 Pa	=	1 N m^{-2}
frequency	hertz (Hz)	1 Hz	=	1 s^{-1}
charge	coulomb (C)	1 C	=	1 A s
potential difference	volt (V)	1 V	=	1 J C^{-1}
electric field	N C^{-1}	1 N C^{-1}	=	1 V m^{-1}
resistance	ohm (Ω)	1 Ω	=	1 V A^{-1}
capacitance	farad (F)	1 F	=	1 A s V^{-1}
inductance	henry (H)	1 H	=	1 V s A^{-1}
magnetic field	tesla (T)	1 T	=	1 N s m^{-1} C^{-1} = 1 kg s^{-2} A^{-1}

SI base units and multiples

The units used in this course are SI (standing for Système International d'Unités). In SI, there are seven base units, as listed below.

Physical quantity	Name of unit	Symbol of unit
Length	metre	m
mass	kilogram	kg
time	second	s
electric current	ampere	A
temperature	kelvin	K
luminous intensity	candela	cd
amount of substance	mole	mol

The most commonly used standard SI multiples and submultiples are given below.

Multiple	Prefix	Symbol for prefix	Submultiple	Prefix	Symbol for prefix
10^{12}	tera	T	10^{-3}	milli	m
10^{9}	giga	G	10^{-6}	micro	μ
10^{6}	mega	M	10^{-9}	nano	n
10^{3}	kilo	k	10^{-12}	pico	p
10^{0}	—	—	10^{-15}	femto	f

Useful conversions

1 degree ≈ 0.01745 radian; 1 radian ≈ 57.30°

absolute zero: 0 K = –273.15 °C   1 electronvolt (eV) = 1.602×10^{-19} J

Useful constants

magnitude of the acceleration due to gravity (on Earth)	g	=	$9.81\ \mathrm{m\,s^{-2}}$
Newton's universal gravitational constant	G	=	$6.673 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Avogadro's constant	N_m	=	$6.023 \times 10^{23}\ \mathrm{mol^{-1}}$
Boltzmann's constant	k	=	$1.381 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
molar gas constant	R	=	$8.314\ \mathrm{J\,K^{-1}\,mol^{-1}}$
permittivity of free space	ε_0	=	$8.854 \times 10^{-12}\ \mathrm{C^2\,N^{-1}\,m^{-2}}$
	$1/4\pi\varepsilon_0$	=	$8.988 \times 10^{9}\ \mathrm{N\,m^2\,C^{-2}}$
permeability of free space	μ_0	=	$4\pi \times 10^{-7}\ \mathrm{T\,m\,A^{-1}}$
speed of light in vacuum	c	=	$2.998 \times 10^{8}\ \mathrm{m\,s^{-1}}$
Planck's constant	h	=	$6.626 \times 10^{-34}\ \mathrm{J\,s}$
	$\hbar = h/2\pi$	=	$1.055 \times 10^{-34}\ \mathrm{J\,s}$
Rydberg constant	R	=	$1.097 \times 10^{7}\ \mathrm{m^{-1}}$
Bohr radius	a_0	=	$5.292 \times 10^{-11}\ \mathrm{m}$
atomic mass unit	amu (or u)	=	$1.6603 \times 10^{-27}\ \mathrm{kg}$
charge of proton	e	=	$1.602 \times 10^{-19}\ \mathrm{C}$
charge of electron	$-e$	=	$-1.602 \times 10^{-19}\ \mathrm{C}$
electron rest mass	m_e	=	$9.109 \times 10^{-31}\ \mathrm{kg}$
charge to mass ratio of the electron	$-e/m_\mathrm{e}$	=	$-1.759 \times 10^{11}\ \mathrm{C\,kg^{-1}}$
proton rest mass	m_p	=	$1.673 \times 10^{-27}\ \mathrm{kg}$
neutron rest mass	m_n	=	$1.675 \times 10^{-27}\ \mathrm{kg}$
radius of the Earth			$6.378 \times 10^{6}\ \mathrm{m}$
mass of the Earth			$5.977 \times 10^{24}\ \mathrm{kg}$
mass of the Moon			$7.35 \times 10^{22}\ \mathrm{kg}$
mass of the Sun			$1.99 \times 10^{30}\ \mathrm{kg}$
average radius of Earth orbit			$1.50 \times 10^{11}\ \mathrm{m}$
average radius of Moon orbit			$3.84 \times 10^{8}\ \mathrm{m}$

The Greek alphabet

(pronunciation is given in brackets where not obvious)

alpha	A	α	eta	H	η	nu (new)	N	ν	tau (taw)	T	τ
beta	B	β	theta	Θ	θ, ϑ	xi (csi)	Ξ	ξ	upsilon	Y	υ
gamma	Γ	γ	iota	I	ι	omicron	O	o	phi (fie)	Φ	ϕ
delta	Δ	δ	kappa	K	κ	pi (pie)	Π	π	chi (kie)	X	χ
epsilon	E	ε	lambda	Λ	λ	rho (roe)	P	ρ	psi	Ψ	ψ
zeta	Z	ζ	mu (mew)	M	μ	sigma	Σ	σ	omega	Ω	ω

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